Cellular Automata and comonads: An overview

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Overview

- Cellular automata (CA) are synchronous distributed systems where the next state of each device only depends on the current state of its neighbors.
- Their implementation on a computer is straightforward, making them very good tools for simulation and qualitative analysis.
- We re-interpret them using category theory.
- We retrieve classical results on CA as special cases of general facts in category theory.
- We suggest further directions to explore.

Cellular automata

A cellular automaton (CA) on a monoid G is a triple $\mathcal{A} = \langle A, \mathcal{N}, d \rangle$ where:

- A is a finite alphabet;
- $\mathcal{N} = \{n_1, \ldots, n_m\} \subseteq G$ is a finite neighborhood index
- $d : A^m \to A$ is a finitary transition function

Local and global behavior of CA

Let $G = (G, 1_G, \cdot)$ a monoid, $\mathcal{A} = \langle A, \mathcal{N}, d \rangle$ a CA on G

- G induces on the *configurations* $c \in A^G$ a family of translations

$$c^g = \sigma_g(c) = \lambda h : G.c(g \cdot h)$$

- $\mathcal{A}$ induces a *local behavior* $\Lambda_{\mathcal{A}}(c) : A^G \to A$ by

$$\begin{aligned} \Lambda_{\mathcal{A}}(c) &= f\left(c|_{\mathcal{N}}\right) \\ &= f\left(c(n_1), \ldots, c(n_m)\right) \end{aligned}$$

- $\mathcal{A}$ induces a *global behavior* $\Gamma_{\mathcal{A}}(c) : A^G \to A^G$ by

$$\begin{aligned} \Gamma_{\mathcal{A}}(c) &= \lambda g : G.f\left(c^g|_{\mathcal{N}}\right) \\ &= \lambda g : G.f\left(c(g \cdot n_1), \ldots, c(g \cdot n_m)\right) \end{aligned}$$

Two classical results

Curtis-Lyndon-Hedlund theorem

Let $f : A^G \to A^G$. The following are equivalent.

1. f is the global behavior of a CA.
2. f is continuous in the product topology and commutes with the translations.

Reason why: compactness of A^G and uniform continuity of f.

Reversibility principle

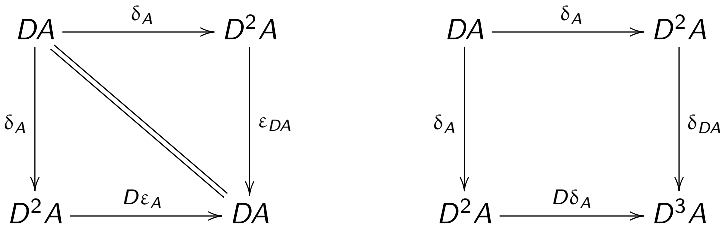
- Let f be a bijective CA global behavior.
- Then f^{-1} is also the global behavior of some CA.

Reason why: f is a homeomorphism + Curtis-Hedlund.

Comonads

A *comonad* on a category $\mathcal{C}$ is a triple $D = (D, \varepsilon, \delta)$ where:

- D is a functor from $\mathcal{C}$ to itself;
- the counit $\varepsilon : D \to \mathrm{Id}_{\mathcal{C}}$ and the *comultiplication* $\delta : D \to D^2$ are natural transformations;
- for every $A \in |\mathcal{C}|$ the following diagrams commute:



Equivalently: a comonad on $\mathcal{C}$ is a monad on $\mathcal{C}^{\mathrm{op}}$.

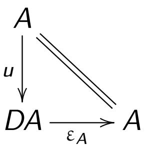
Why comonads?

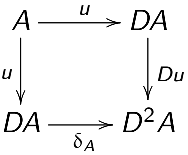
1. Comonads provide a solution to the general problem of finding an adjunction generating an endofunctor.
2. Comonads appear "naturally" in context-dependent computation.
3. Comonads also appear to be "natural" models for "emergent" computation—such as CA.

Coalgebras on a comonad

Let $D = (D, \varepsilon, \delta)$ a comonad on a category $\mathcal{C}$.

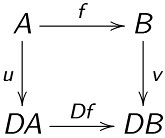
- A *D-coalgebra* is a pair (A, u), $A \in |\mathcal{C}|$, $u \in \mathcal{C}(A, DA)$ such that the following diagram commutes:





  Note that (DA, δ_A) is always a coalgebra, thanks to the comonad laws; we call these *cofree coalgebras*.

- A *coalgebra morphism* from (A, u) to (B, v) is an $f \in \mathcal{C}(A, B)$ such that the following diagram commutes:



Constructions on comonads

The coKleisli category $\mathrm{coKl}(D)$

- Objects: $|\mathrm{coKl}(D)| = |\mathcal{C}|$.
- Maps: $\mathrm{coKl}(D)(A, B) = \mathcal{C}(DA, B)$.
- Identities: $\mathrm{jd}_A = \varepsilon_A$.
- Composition: $g \bullet f = g \circ f^\dagger$ where $f^\dagger = Df \circ \delta_A$.

The coEilenberg-Moore category $\mathrm{coEM}(D)$

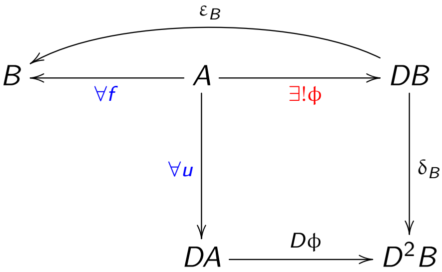
- Objects: D-coalgebras.
- Maps: coalgebra morphisms.
- Identities and composition: same as in $\mathcal{C}$.

The Key Fact

Theorem (dual classical)

$\mathrm{coKl}(D)$ is equivalent to the full subcategory of $\mathrm{coEM}(D)$ generated by the cofree coalgebras.

The trick is that the cofree coalgebras are *final objects* in $\mathrm{coEM}(D)$:



Uniform spaces

A *uniform space* is a set X together with a uniformity $\mathcal{U}$ made of entourages $U \subseteq X \times X$ such that:

1. For every $U \in \mathcal{U}$, $\Delta = \{(x, x) \mid x \in X\} \subseteq U$.
2. If $U \subseteq V$ and $U \in \mathcal{U}$ then $V \in \mathcal{U}$.
3. If $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$.
4. If $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$.
5. If $U \in \mathcal{U}$ then $\exists V \in \mathcal{U} \mid V^2 \subseteq U$.

The richest uniformity is the *discrete uniformity*

$$\mathcal{D} = \{U \subseteq X \times X \mid \Delta \subseteq U\}$$

Uniformities and topologies

Uniform spaces are "between" topological and metric spaces:

- If $\mathcal{U}$ is a uniformity on X, then the family of the sets:

$$U[x] = \{y \in X \mid (x, y) \in U\}, \quad x \in X, U \in \mathcal{U}$$

  is a *basis* for a topology $\mathcal{T}$ on X, that is, every element of $\mathcal{T}$ is a union of sets of the form $U[x]$.
- If d is a distance on X, then the family $\mathcal{U} = \{U_\delta\}_{\delta > 0}$ where:

$$U_\delta = \{(x, y) \in X \times X \mid d(x, y) \leqslant \delta\} = d^{-1}([0, \delta])$$

  is a uniformity on X.

The discrete uniformity induces the discrete topology.
However, there *exist* nondiscrete uniformities which induce the discrete topology!

The category $\mathbf{Unif}$ of uniform spaces

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces.
$f : X \to Y$ is *uniformly continuous* (briefly, u.c.) if it satisfies one of the following, equivalent, conditions:

1. For every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $(f \times f)(U) \subseteq V$.
2. $(f \times f)^{-1}(V) \in \mathcal{U}$ for every $V \in \mathcal{V}$.

The following are immediate:

- The identity is uniformly continuous.
- Composition of u.c. functions is u.c.

We denote by $\mathbf{Unif}$ the category of uniform spaces with u.c. functions.

Currying back and forth

Definition in $\mathbf{Set}$:

- Let $f : A \times X \to Y$, $\bar{f} : A \to Y^X$ satisfy $f(a, x) = \bar{f}(a)(x)$ $\forall a, x$.
- We call $\bar{f}$ the currying of f, and f the uncurrying of $\bar{f}$.

Issues in $\mathbf{Top}$:

- Y^X must be given a topology, call it $\mathcal{T}$.
- Such $\mathcal{T}$ must make f continuous if and only if so is $\bar{f}$.
- It turns out that $\mathcal{T}$ is either nonexistent, or unique.
- For X discrete, $\mathcal{T}$ is the *product topology*: coarsest making projections continuous.

Issues in $\mathbf{Unif}$:

- Same as in $\mathbf{Top}$, with uniformities in place of topologies.
- If X has the discrete uniformity, then Y^X may be given the product uniformity, *i.e.*, coarsest making evaluations u.c.

Local behaviors on uniform spaces

Let $G = (G, 1_G, \cdot)$ be a uniformly discrete monoid.

Definition

A *local behavior* between two uniform spaces A, B is a uniformly continuous function

$$k : A^G \longrightarrow B$$

where A^G is *prodiscrete* (product of discrete uniformities).

The rationale for this:

- A CA local behavior k derives from a finitary function d.
- Curtis-Lyndon-Hedlund: k is u.c. with A discrete and A^G prodiscrete.

Global behaviors on uniform spaces

Let $G = (G, 1_G, \cdot)$ be a uniformly discrete monoid.

Definition

The *global behavior* associated to a local behavior $k : A^G \to B$ is the uniformly continuous function $k^\dagger : A^G \to B^G$ defined by:

$$k^\dagger(c) =_{\mathrm{df}} \lambda x : G.k(c \triangleright x)$$

where

$$c \triangleright x =_{\mathrm{df}} \lambda y : G.c(x \cdot y)$$

The rationale for this:

- A CA global behavior derives from application of local behavior to translates.

The category of local behaviors

Definition

- Objects: uniform spaces.
- Maps: local behaviors.
- Identities: $\mathrm{jd}_A(c) =_{\mathrm{df}} c(1_G)$.
- Compositions: $\ell \bullet k = \ell \circ k^\dagger$.

This looks like the coKleisli category of some comonad on $\mathbf{Unif}$...

The exponent comonad

Definition

A uniformly discrete monoid G determines a comonad $D = (D, \varepsilon, \delta)$ on $\mathbf{Unif}$ as follows:

- $DA =_{\mathrm{df}} A^G$ with product uniformity for $A \in |\mathbf{Unif}|$.
- $Df =_{\mathrm{df}} \lambda c : A^G . f \circ c$ for $f \in \mathbf{Unif}(A, B)$.
- $\varepsilon_A =_{\mathrm{df}} \lambda c : A^G . c(1_G)$ for $A \in |\mathbf{Unif}|$.
- $\delta_A c =_{\mathrm{df}} \lambda x : G . c \triangleright x$ for $c \in \mathbf{Unif}(A^G, B^G)$.

Then:

- Local behaviors are the maps of $\mathrm{coKl}(D)$!
- Global behaviors are the cofree coalgebra maps for D!

Interpretation of coalgebras

Given u, let $a \otimes x = u(a)(x)$. Then:

$$\begin{array}{ccc} A & \xrightarrow{u} & A^G \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon_A} \\ & & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{u} & A^G \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \delta_A} \\ A^G & \xrightarrow{u^G} & (A^G)^G \end{array}$$

become: (with some unital and associative laws in the middle)

$$\begin{array}{ccc} A & \xrightarrow{A \times 1_G} & A \times G \\ & \searrow^{=} & \downarrow{\scriptstyle \otimes} \\ & & A \end{array} \qquad \begin{array}{ccc} A \times G \times G & \xrightarrow{A \times (\cdot)} & A \times G \\ {\scriptstyle \otimes \times G}\downarrow & & \downarrow{\scriptstyle \otimes} \\ A \times G & \xrightarrow{\otimes} & A \end{array}$$

Thus:

the D-coalgebras are the curryings
of the (uniformly continuous) *actions* of G.

Interpretation of coalgebra morphisms

Let $\otimes$ and $\oslash$ be the uncurryings of u and v, respectively. Then:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A^G & \xrightarrow{f^G} & B^G \end{array} \quad \text{becomes} \quad \begin{array}{ccc} A \times G & \xrightarrow{f \times \mathrm{id}_G} & B \times G \\ {\scriptstyle \otimes}\downarrow & & \downarrow{\scriptstyle \oslash} \\ A & \xrightarrow{f} & B \end{array}$$

Thus

the D-coalgebra morphisms are the maps
that commute with the respective actions

in the sense that

$$f(a \otimes x) = f(a) \oslash x$$

Interpretation of cofree coalgebra morphisms

If u and v are δ_A and δ_B, then:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle \delta_B} \\ A^G & \xrightarrow{f^G} & B^G \end{array} \quad \text{becomes} \quad \begin{array}{ccc} A^G \times G & \xrightarrow{f \times \mathrm{id}_G} & B^G \times G \\ {\scriptstyle \triangleright_A}\downarrow & & \downarrow{\scriptstyle \triangleright_B} \\ A^G & \xrightarrow{f} & B^G \end{array}$$

which yields:

$$f(c \triangleright_A x) = f(c) \triangleright_B x$$

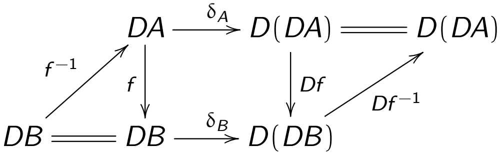
But $\triangleright$ is the translation ($=$ cofree action). We thus retrieve:

the cofree coalgebra morphisms
are the translation-commuting maps

Reversible global behaviors

Let $f : X \to Y$ be uniformly continuous.

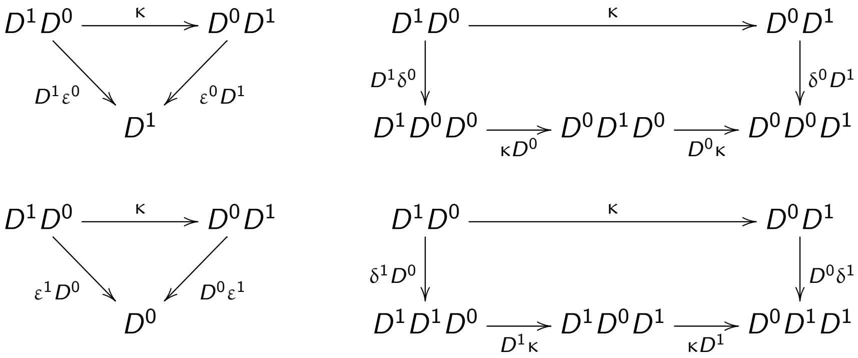
- Even if f is bijective, f^{-1} needs not be u.c.
- This, however, is ensured if X is compact (with the induced topology).
- Now, if A is discrete, then A^G is compact iff A is finite.
- But for any comonad D on any category $\mathcal{C}$, if the inverse of a coalgebra morphism is in $\mathcal{C}$, then it is a coalgebra morphism:



The reversibility principle is thus an instance of this general fact.

Distributive laws: Definition

Let two comonads $D^i = (D^i, \varepsilon^i, \delta^i)$ be given.
A *distributive law* of D^1 over D^0 is a natural transformation $\kappa : D^1 D^0 \to D^0 D^1$ such that the following diagrams commute:



Distributive laws: Meaning

Distributive laws allow *composing comonads into comonads:*

- A distributive law induces a comonad

$$D = (D^1 D^0, \varepsilon^1 \varepsilon^0, \delta^1 \kappa \delta^0)$$

- The comonad D^0 *lifts* to a comonad $\overline{D^0}$ *over* $\mathrm{coKl}(D^1)$ defined by:
  - $\overline{D^0} A =_{\mathrm{df}} A$;
  - $\overline{D^0} f =_{\mathrm{df}} f \circ \kappa_A : D^1 D^0 A \to D^0 B$;
  - $\overline{\varepsilon^0}_A =_{\mathrm{df}} \varepsilon^0_A \circ \varepsilon^1_{D_0 A}$;
  - $\overline{\delta^0}_A =_{\mathrm{df}} \delta^0_A \circ \varepsilon^1_{D^0 A}$.

Many dimensions

Idea:

- Suppose we have *two* monoids G_0, G_1.
- There is a natural isomorphism $(A^{G_0})^{G_1} \cong A^{G_0 \times G_1}$.
- We can then think of a $k : (A^{G_0})^{G_1} \longrightarrow A$ either:
  - As a *2D CA* on $G_0 \times G_1$ between A and B.
  - As a 1D CA on G_1 between A^{G_0} and B.

Realization:

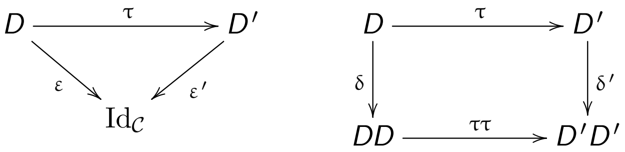
- Let G^i define comonad D^i. The following is a distributive law:

$$\kappa_A(c)(x_1)(x_0) =_{\mathrm{df}} c(x_0)(x_1)$$

- $DA =_{\mathrm{df}} (A^{G_0})^{G_1}$ is a comonad and $\mathrm{coKl}(D) = \mathrm{coKl}(\overline{D^0})$.
- So k can be seen as a $\mathrm{coKl}(D^1)$-CA on G_0 from A to B.

Comonad maps

A *comonad map* from D to D' is a natural transformation $\tau : D \to D'$ such that the following diagrams commute:



Meaning:

- Comonad maps preserve counits and comultiplications.
- Comonads and comonad maps form a category.

Point-dependent behavior

In addition to our comonad D, consider the following comonad D':

- $D'A =_{\mathrm{df}} A^G \times G$;
- $D'f =_{\mathrm{df}} (f \circ -) \times \mathrm{id}_G$ for $f : A \to B$;
- $\varepsilon'_A(c, x) =_{\mathrm{df}} c(x)$ for $c \in A^G$;
- $\delta'_A(c, x) = (\lambda y.(c, y), x)$ for $c \in A^G$

Then:

- D'-local behaviors satisfy $k^\dagger(c, x) = (\lambda y.k(c, y), x)$.
- D'-global behaviors satisfy $f(c, x) = (g(c), x)$ for some g.
- The translation $\triangleright$ is a comonad map from D' to D.
- "Ordinary" local behaviors are point-dependent local behaviors that don't take the point into account.

A possible future direction

Call *asymptotic* two configurations which differ at most in finitely many points.

A discrete group G is *amenable* if it has a *mean* $m : \ell_\infty(G) \to \mathbb{R}$ which is:

1. linear;
2. *nonnegative:* if $f(x) \geqslant 0$ for every $x \in G$ then $m(f) \geqslant 0$;
3. *consistent:* $m(\lambda x.1) = 1$.

- A CA is *pre-injective* if every two different asymptotic configurations have different images.
- *Garden of Eden theorem (Moore and Myhill, 1962):* A CA on $G = \mathbb{Z}^d$ is pre-injective if and only if it is surjective.
- *Bartholdi, 2010:* The Garden of Eden theorem holds for CA on G if and only if G is amenable.

How to obtain the Garden of Eden theorem in our setting?

Conclusions

We set up an experiment with definitions.

- We have checked that CA arise as "natural" constructions with "natural" properties.
- We have retrieved some classical results as instances of general facts.
- We have checked further developments of this point of view.

We confidently say that the experiment has succeeded.

Thank you for attention!

Any questions?