CATEGORY THEORY ITI9200 EXERCISES

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How to approach these exercises

- The ability to think is rewarded more than correctness; this means that good ideas leading to wrong answers are more valuable than bad ideas yielding the correct answer. Category theory is founded on the belief that the right answer is useless when found by means of an unenlightening train of thought.
- Keep in mind that every exercise has just a finite number of correct answers, and an infinite number of wrong answers (wink wink).
- Every exercise is marked with a certain number of ☕'s, according to the Rényi-Erdös complexity scale:
  - C1) ☕ : this is an exercise that merely requires to sit down, think, and solve the puzzle, helped by a cup of coffee; you are supposed to be able to solve the 1-cup exercises.
  - C2) ☕☕ : this is an exercise that requires a cozy spot in a library, silence, and a little more care in the choice of coffee (American coffee proved to be an insufficient adjuvant); you are supposed to be able to solve the 2-cup exercises, with a little help (that *may* be coming from the exercise sessions, wink wink).
  - C3) ☕☕☕ : these are usually the optional exercises. They are supposed to be difficult: don't be put off, enjoy the process of discovery, helped by your favourite psychotropic drug.
- Give the optional exercises a try: a failed attempt based on a good idea will be rewarded.

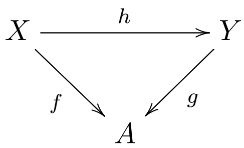
1. Categories, functors, natural transformations

> Primer ejercicio es meditación con las tres potencias sobre el primero, segundo y tercer pecado.

Exercise 1

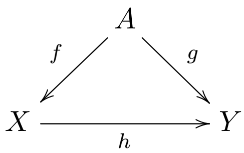
This exercise studies how the constructions of opposite category, slice, coslice, product, and arrow category interact with each other: recall that the slice category $\mathcal{C}/A$ of objects over A has objects the arrows of $\mathcal{C}$ with fixed codomain A, and morphisms suitable commutative triangles:

(1.1)



The *coslice* category $A/\mathcal{C}$ has objects the arrows of $\mathcal{C}$ having fixed *domain* A, and morphisms suitable commutative triangles:

(1.2)



The arrow category $\mathcal{C}^{\to}$ has objects the arrows of $\mathcal{C}$, and morphisms suitable commutative squares.

Given a category $\mathcal{A}$, we denote the opposite category as $\mathcal{A}^o$. Given all this, prove or disprove the following equalities

- for every category $\mathcal{C}$, $(\mathcal{C}^{\to})^o = (\mathcal{C}^o)^{\to}$
- for every category $\mathcal{C}$, $\mathcal{C}/A = (A/\mathcal{C})^o$
- for categories $\mathcal{C}, \mathcal{D}$, $(\mathcal{C} \times \mathcal{D}^o) = \mathcal{C}^o \times \mathcal{D}$
- If $f : A \to B$, $(\mathcal{C}/B)/f = \mathcal{C}/A$. (Regarding this last item, first convince yourself that $(\mathcal{C}/B)/f$ is well defined, in that f is an object of the category $\mathcal{C}/B$.)

Exercise 2

Let $\mathrm{P} : \mathsf{Set} \to \mathsf{Set}$ be the correspondence that sends a set A to the *power set* $\mathrm{P}A$ of A, the set of all subsets $U \subseteq A$, and a function $f : A \to B$ to the function

$Pf : \mathrm{P}A \to PB$, that sends a subset $U \subseteq A$ to the *image*

$$f_* U := \{ fu \mid u \in U \} \tag{1.3}$$

Similarly, let $\rotatebox[origin=c]{180}{P} : \mathsf{Set} \to \mathsf{Set}$ be the correspondence that sends A to $\mathrm{P}A$, but a function $f : A \to B$ to the function $PB \to \mathrm{P}A$, that sends a subset $V \subseteq B$ to the *inverse image*

$$f^* V := \{ a \in A \mid fa \in V \} \tag{1.4}$$

- Show that both $\mathrm{P}, \rotatebox[origin=c]{180}{P}$ are functors ($\rotatebox[origin=c]{180}{P}$ is contravariant, i.e. $\rotatebox[origin=c]{180}{P}(f \circ g) = \rotatebox[origin=c]{180}{P} g \circ \rotatebox[origin=c]{180}{P} f$); show that given subsets $U \in \mathrm{P}A, V \in \mathrm{P}B$ one has

$$f_* U \subseteq V \iff U \subseteq f^* V. \tag{1.5}$$

- What is a natural transformation $f^* f_* \Rightarrow 1_{\mathrm{P}A}$, regarding $\mathrm{P}A$ as a category? Show that for each $U \in \mathrm{P}A$, one has $U \subseteq f^* f_* U$: is this a natural transformation $f^* f_* \Rightarrow 1_{\mathrm{P}A}$?

Exercise 3

From the category of sets, remove all functions $A \to B$ when $A \neq B$ are different sets; is the result still a category $\mathcal{C}$?

Define a correspondence $G : \mathcal{C} \to \mathsf{Mon}$ (the category of monoids), sending a set A to the monoid $A^\star$ of finite lists of elements of A, i.e. to the set of all finite lists $(a_1, \dots, a_n)$ where $n \geq 0$ and $a_i \in A$ for each $i = 1, \dots, n$, and a function $f : A \to A$ to the function

$$A^\star \to A^\star : (a_1, \dots, a_n) \mapsto (f(fa_1), \dots, f(fa_n)) \tag{1.6}$$

Is G a functor $\mathcal{C} \to \mathsf{Mon}$?

Exercise 4 (*This exercise is optional*)

Define a category $\mathcal{C}$ with a single object $\bullet$, and where the set of morphisms $\bullet \to \bullet$ is specified in BNF as

$$t ::= x_0 \mid c \mid f\, t \mid g\, t \tag{1.7}$$

where x_0 is one given variable, c is a constant and f, g are two *different* given function symbols. Composition is defined as substitution $t[t'/x_0]$ (where t' replaces x_0 in t is defined recursively).

If you know what to do with it, you are allowed to use a proof-assistant that checks the axioms of category. You could e.g. define

```
-- the category
data C : Set where
  x₀  : C
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    c   : C
    f g : C → C
  -- the composition law
  _∘_ : C → C → C
  s ∘ t = ?
```

and from here provide proofs that ∘ is associative and that C has an identity morphism:

```
Associative : {A : Set} → (A → A → A) → Set
Associative {A} u =
  ∀ { x y z : A } → u (u x y) z ≡ u x (u y z)

Identityʳ : {A : Set} → (A → A → A) → A → Set
Identityʳ {A} u e = ∀ {x : A} → u e x ≡ x

Identityˡ : {A : Set} → (A → A → A) → A → Set
Identityˡ {A} u e = ∀ {x : A} → u x e ≡ x
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If you are very brave: a monoid is precisely a category with a single object; this means that $\mathcal{C}$ above is isomorphic to a monoid, whose elements are the terms $t = x_0 \mid c \mid f\,t \mid g\,t$ and whose monoid operation is defined by substitution. Describe the monoid M.