CATEGORY THEORY ITI9200 EXERCISES

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How to approach these exercises

- The ability to think is rewarded more than correctness; this means that good ideas leading to wrong answers are more valuable than bad ideas yielding the correct answer. Category theory is founded on the belief that the right answer is useless when found by means of an unenlightening train of thought.
- Keep in mind that every exercise has just a finite number of correct answers, and an infinite number of wrong answers (wink wink).
- Every exercise is marked with a certain number of ☕'s, according to the Rényi-Erdös complexity scale:
  - C1) ☕ : this is an exercise that merely requires to sit down, think, and solve the puzzle, helped by a cup of coffee; you are supposed to be able to solve the 1-cup exercises.
  - C2) ☕☕ : this is an exercise that requires a cozy spot in a library, silence, and a little more care in the choice of coffee (American coffee proved to be an insufficient adjuvant); you are supposed to be able to solve the 2-cup exercises, with a little help (that *may* be coming from the exercise sessions, wink wink).
  - C3) ☕☕☕ : these are usually the optional exercises. They are supposed to be difficult: don't be put off, enjoy the process of discovery, helped by your favourite psychotropic drug.
- Give the optional exercises a try: a failed attempt based on a good idea will be rewarded.

1. Universal properties, Limits and colimits

> Segundo ejercicio es meditacion de los pecados, y contiene en si, despues de la oracion preparatoria y dos preámbulos, cinco puntos y un coloquio.

Exercise 1

Let $1 = \{\bot\}$ be any set with a single element $\bot$; consider the correspondence that sends a set A into $A \sqcup 1$, i.e. to the coproduct of A and 1.

This is a functor $\_ \sqcup 1$ from the category Set of sets and functions to itself, called the "*add a disjoint point to a set*", or the "*maybe*" construction (for reasons rooted in functional programming).

Show that $\_ \sqcup 1$ is indeed a functor.

Show that the following diagram commutes:

$$\begin{array}{ccc} 1 \sqcup 1 \sqcup 1 & \xrightarrow{1 \sqcup \mathsf{fold}} & 1 \sqcup 1 \\ {\scriptstyle \mathsf{fold} \sqcup 1}\downarrow & & \downarrow{\scriptstyle \mathsf{fold}} \\ 1 \sqcup 1 & \xrightarrow[\mathsf{fold}]{} & 1 \end{array} \tag{1.1}$$

where $\mathsf{fold} : 1 \sqcup 1 \to 1$ is the unique function from $1 \sqcup 1$ to the point, since 1 is terminal in Set; fold is also obtained from the universal property of the coproduct: in what way?

Show that the universal property of the coproduct yields a unique pair of functions (i_A, i_1), $i_A : A \to A \sqcup 1$, $i_1 : 1 \to A \sqcup 1$.

Show that the set of functions $a : A \sqcup 1 \to A$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{i_A} & A \sqcup 1 \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} \tag{1.2}$$

commutes is in bijection with the elements of A; this means that each such function $a : A \sqcup 1 \to A$ determines a unique element $a_0 : A$.

Answer of exercise 1

Let's show the functoriality rules for the correspondence $M : A \mapsto A \sqcup 1$: a function $f : A \to B$ between sets goes to the function $Mf : MA \to MB$,

$$f \sqcup 1 : A \sqcup 1 \to B \sqcup 1$$

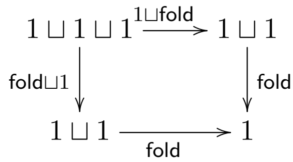
Evidently, Mf can be defined separately on A and on 1; we posit that on A, Mf acts as f, sending $a : A$ into $fa : B$, and then embedding $B \hookrightarrow MB$ using the universal property of coproducts. On 1, Mf sends the unique element $\bot$ into $\bot$.

With this definition, the identity function $A \to A$ goes to the identity function $MA \to MA$, and the composition $Mf \circ Mg$ is exactly $M(f \circ g)$, because

$$MA \xrightarrow{Mg} MB \xrightarrow{Mf} MC$$

sends $a : A$ into $f(ga) : C \hookrightarrow MC$ and $\bot$ to $\bot$.

Let's now consider the diagram



and before that, the fold map: the universal property of the coproduct establised a bijection

$$\mathsf{Set}(1 \sqcup 1, 1) \cong \mathsf{Set}(1, 1) \times \mathsf{Set}(1, 1).$$

On the right hand side there is a product of one-element hom-sets; the corresponding element of the hom-set on the left is fold, the unique map $1 \sqcup 1 \to 1$. Moreover, the diagram above must commute, because the codomain of both $\mathsf{fold} \circ (1 \sqcup \mathsf{fold})$ and $\mathsf{fold} \circ (\mathsf{fold} \sqcup 1)$ is a terminal object of Set.

Using the universal property of the coproduct $MA = A \sqcup 1$, we get a bijection

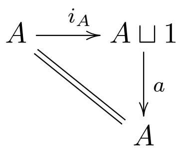
$$\mathsf{Set}(MA, MA) \cong \mathsf{Set}(A, MA)\mathsf{Set}(1, MA)$$

On the left hand side, there is for sure at least the identity function id_{MA}; under the bijection, id_{MA} corresponds to a pair of functions

$$A \xrightarrow{i_A} MA \xleftarrow{i_1} 1$$

with the property that for every pair $j_A, j_1 : A, 1 \to X$, there is a unique $j : MA \to X$ such that... This is exactly the universal property of the coproduct, just written in terms of a bijection on hom-sets.

Let's now consider the diagram



and in particular a function $a : MA \to A$ with the property that $a \circ i_A$ is the identity on A; the action on MA of a can be determined with a case split on $x : MA$, because $MA = A \sqcup 1$

- if $x : MA$ is an element of A, i.e. $x = x' : A$, the condition $a \circ i_A$ entails that $a(x) = x$.
- if $x : MA$ is $\perp$, there is no condition on the element $a(\perp)$, so it can be an arbitrary element $a_0 : A$.

Evidently, every $a_0 : A$ determines a function $a : MA \to A$ as above, thus establishing the desired bijection.

☕ Exercise 2

Let $\mathcal{C} = (\mathbb{N}, \geq)$ be the category having objects the natural numbers $0, 1, 2, \dots$ and where there is a morphism $m \to n$ if and only if $n \leq m$ (read twice this definition). Does $\mathcal{C}$ have a terminal object? Does it have an initial object? Does it have products $n \times m$? Does it have coproducts $n \sqcup m$? Answer the same four questions for the category $\mathcal{D} = (\mathbb{N}, \_ \mid \_)$, that has the same objects and where there is a morphism $n \to m$ if and only if $m = kn$ for some $k \in \mathbb{N}$ (the relation $n \mid m$ reads as "n divides m").

Answer of exercise 2

$\mathcal{C}$ is the category

$$\{\dots \leq 3 \leq 2 \leq 1 \leq 0\}$$

So the poset $(\mathbb{N}, \leq^{\mathrm{op}})$ of natural numbers with the opposite order; in short, $\mathcal{C} = (\mathbb{N}, \leq)^{\mathrm{op}}$. So, a categorical thingy in $\mathcal{C}$ exists if and only if the co-thingy exists in $(\mathbb{N}, \leq)$. Given this, it's easy to answer each question!

A terminal object in a poset regarded as a category is just a *top element*; in $\mathcal{C}$, 0 is the top element (because it is a bottom element, i.e. initial, in $(\mathbb{N}, \leq)$).

There is no initial object in $\mathcal{C}$, because this would mean that $(\mathbb{N}, \leq)$ has a terminal object, i.e. a top element: contrary to the belief of some, there is no natural number bigger than all others.

Given two natural numbers p, q in $(\mathbb{N}, \leq)$, their product in $(\mathbb{N}, \leq)$ coincides with their minimum, and their coproduct with their maximum: so, $p \times_{\mathcal{C}} q = \max\{p, q\}$, and $p \sqcup_{\mathcal{C}} q = \min\{p, q\}$.

As a side note, this extends to infinite coproducts in $\mathcal{C}$, but not to infinite products (say, the set of prime numbers has no maximal element in $(\mathbb{N}, \leq)$, so there's no minimal element for that family in $\mathcal{C}$).

Now, let's answer to the same questions for $\mathcal{D}$: the relation here is divisibility of natural numbers; a minimal element is thus 1 (for every $m \in \mathbb{N}$ there exists $h = m \in \mathbb{N}$ such that $m = m \cdot 1$), and a maximal element in 0 (dual of the above).

Minimal elements of $\mathcal{D}$ are prime numbers, in the sense that if p is prime and morphisms $1 \to j \to p$ are given, then at least one is an identity.

Products and coproducts correspond to lcm and gcd of two objects.

Exercise 3

Let $\mathcal{C}, \mathcal{D}, \mathcal{Z}$ be three categories, and $\mathcal{C} \xrightarrow{F} \mathcal{Z} \xleftarrow{G} \mathcal{D}$ two functors; define the *comma category* of the functors F, G to be the category (F/G) whose

- objects are arrows in $\mathcal{Z}$ of the form $f : FC \to GD$ for a pair of objects $C \in \mathcal{C}, D \in \mathcal{D}$ (more formally, an object of (F/G) is a tuple $(C, D, f : Fc \to Gd)$);
- morphisms $(C, D, f) \to (C', D', f')$ are pairs $u : C \to C'$, $v : D \to D'$ with the property that the square

(1.3)
$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ {\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle Gv} \\ FC' & \xrightarrow[f']{} & GD' \end{array}$$

is commutative.

- Show that (F/G) is indeed a category defining its composition rule and identity maps.
- What is (F/G) if G is a constant functor (say, at an object Z)? What if G is constant *and* F is the identity functor of $\mathcal{Z}$? What if F, G are *both* constant (say at objects Z, Z')?
- Show that (F/G) has the following universal property: there exists a pair of functors $\mathcal{C} \times \mathcal{D} \xleftarrow{Q} (F/G) \xrightarrow{P} \mathcal{Z}^{\to}$ (the arrow category of $\mathcal{Z}$) with the property that the square

(1.4)
$$\begin{array}{ccc} (F/G) & \xrightarrow{P} & \mathcal{Z}^{\to} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}\text{source}\\ \text{target}\end{smallmatrix}\right]} \\ \mathcal{C} \times \mathcal{D} & \xrightarrow[F \times G]{} & \mathcal{Z} \times \mathcal{Z} \end{array}$$

is commutative, and for every other commutative square

(1.5)
$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{K} & \mathcal{Z}^{\to} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}\text{source}\\ \text{target}\end{smallmatrix}\right]} \\ \mathcal{C} \times \mathcal{D} & \xrightarrow[F \times G]{} & \mathcal{Z} \times \mathcal{Z} \end{array}$$

there is a unique functor $\langle H, K\rangle : \mathcal{X} \to (F/G)$ such that $Q \circ \langle H, K\rangle = H$ and $P \circ \langle H, K\rangle = K$. In other words, (F/G) is the pullback of $J = \left[\begin{smallmatrix}\text{source}\\ \text{target}\end{smallmatrix}\right]$ and $F \times G$.

Answer of exercise 3

Identity maps correspond to squares

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ FC & \xrightarrow[f]{} & GD \end{array}$$

that exist because F, G are functors; composition corresponds to the outer rectangle in

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ {\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle Gu} \\ FC' & \xrightarrow{f'} & GD' \\ {\scriptstyle Fu'}\downarrow & & \downarrow{\scriptstyle Gu'} \\ FC'' & \xrightarrow[f'']{} & GD'' \end{array}$$

and again, this makes sense because F, G are functors.

If G is a constant functor, say at an object $Z : \mathcal{Z}$, objects in (F/G) are triples (C, D, f) where f is a morphism $f : FC \to Z$ (so, no constraints on D), and morphisms $(C, D, f) \to (C', D', f')$ are pairs of morphisms $(h : \left[\begin{smallmatrix} FC \\ \downarrow \\ Z\end{smallmatrix}\right] \to \left[\begin{smallmatrix} FC' \\ \downarrow \\ Z\end{smallmatrix}\right], k : D \to D')$ such that h fits into a commutative triangle

$$\begin{array}{ccc} FC & \xrightarrow{h} & FC' \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

and k runs freely over morphisms in $\mathcal{D}$. If in addition F is the identity functor of $\mathcal{Z}$, (F/Z) is the product $\mathcal{C}/Z \times \mathcal{D}$ of the comma category $\mathcal{C}/Z$ of objects over Z, and of $\mathcal{D}$.

If F, G are both constant, say at objects $Z, Z' : \mathcal{Z}$,then (F/G) is a discrete category (a set) having elements the arrows $\mathcal{Z}(Z, Z')$.

The functors P, Q act as 'projections', in the sense that representing an object of (F/G) as a triple (C, D, f), $P(C, D, f) = f : \mathcal{Z}^{\to}$ and $Q(C, D, f) = (C, D) : \mathcal{C} \times \mathcal{D}$. Evidently, this choice of P, Q makes the square commute.

Given this, let's prove the universal property: given a square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{K} & \mathcal{Z}^{\to} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}\text{source}\\ \text{target}\end{smallmatrix}\right]} \\ \mathcal{C}\times\mathcal{D} & \xrightarrow[F\times G]{} & \mathcal{Z}\times\mathcal{Z} \end{array}$$

its commutativity says that the pair of functors H, K is such that, for every $X : \mathcal{X}$, the arrow $KX : \mathcal{Z}^{\to}$ is an arrow $F(H_{\mathcal{C}}X) \to G(H_{\mathcal{D}}X)$, where $H_{\mathcal{C}} : \mathcal{X} \xrightarrow{H} \mathcal{C}\times\mathcal{D} \xrightarrow{\pi_{\mathcal{C}}} \mathcal{C}$, and similarly for $H_{\mathcal{D}}$. More explicitly,

$$\begin{array}{ccc} \text{source}(KX) & = & F(H_{\mathcal{C}}X) \\ {\scriptstyle KX}\downarrow & & \downarrow \\ \text{target}(KX) & = & G(H_{\mathcal{D}}X) \end{array}$$

So one can define

$$\langle H, K\rangle : X \mapsto (H_{\mathcal{C}}X, H_{\mathcal{D}}X, KX)$$

Moreover, this is evidently the only way to define such an $\langle H, K\rangle$, as the 'projections' Q and P force the components of $\langle H, K\rangle$ to be $Q \circ \langle H, K\rangle(X) = (H_{\mathcal{C}}X, H_{\mathcal{D}}X)$ and $P \circ \langle H, K\rangle = KX : \mathcal{Z}^{\to}$.

This last result clarifies the result that $(1_{\mathcal{Z}}/Z) \cong \mathcal{Z}/Z \times \mathcal{D}$: from the fact that pullbacks compose, the diagram

$$\begin{array}{ccccc} \mathcal{Z}/Z\times\mathcal{D} & \longrightarrow & \mathcal{Z}/Z & \longrightarrow & \mathcal{Z}^{\to} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}\text{source}\\ \text{target}\end{smallmatrix}\right]} \\ \mathcal{Z}\times\mathcal{D} & \xrightarrow[\pi]{} & \mathcal{Z}\times\bullet & \xrightarrow[1_{\mathcal{Z}}\times Z]{} & \mathcal{Z}\times\mathcal{Z} \end{array}$$

is a pullback, and exactly $(1_{\mathcal{Z}}/Z)$.

Exercise 4 (*This exercise is optional*)

Let Dyn be the following category:

- Objects are tuples $(X, s; x)$ where $s : X \to X$ is a function on the set X, and $x : X$ is an element;
- Morphisms $(X, s; x) \to (Y, t; y)$ are functions $f : X \to Y$ sending x to y, and such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[t]{} & Y \end{array} \tag{1.6}$$

is commutative.

An object of Dyn is called a *dynamical system* because one can study the 'evolution' of s through time, as the family of iterates $\{\mathrm{id}, s, s^2, s^3, \dots\}$ acting on a point $x : X$.

Show that the object $\mathbf{N} = (\mathbb{N}, s, 0)$ is an initial object of Dyn, where

- $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \dots\}$;
- $s : \mathbb{N} \to \mathbb{N} : \lambda n.n + 1$.

What does the universal property of $\mathbf{N}$ mean in terms of an object $(X, s; x)$?

The *mathematical induction principle* says that

$$\big(Q0 \wedge \textstyle\bigwedge_n \bigwedge_{i \le n} Qi \Rightarrow Q(i+1)\big) \Rightarrow \textstyle\bigwedge_{n:\mathbb{N}} Qn \tag{1.7}$$

(in words, if $Q : \mathbb{N} \to \{0, 1\}$ is a proposition, $Q0$ is true, and $Qn \Rightarrow Q(n+1)$, then Qn is true for all $n : \mathbb{N}$. Is there a way to state the induction principle in terms of the universal property of $\mathbb{N}$?

Hint: Use the universal property of $\mathbb{N}$ to show that if $S \subseteq \mathbb{N}$ is a nonempty subset such that the inclusion $i : S \hookrightarrow \mathbb{N}$ is a morphism in the category Dyn of discrete dynamical systems, then $S = \mathbb{N}$. Deduce the induction principle using a suitable S_Q obtained from the property Q.