CATEGORY THEORY ITI9200 EXERCISES

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How to approach these exercises

- The ability to think is rewarded more than correctness; this means that good ideas leading to wrong answers are more valuable than bad ideas yielding the correct answer. Category theory is founded on the belief that the right answer is useless when found by means of an unenlightening train of thought.
- Keep in mind that every exercise has just a finite number of correct answers, and an infinite number of wrong answers (wink wink).
- Every exercise is marked with a certain number of ☕'s, according to the Rényi-Erdös complexity scale:
  - c1) ☕ : this is an exercise that merely requires to sit down, think, and solve the puzzle, helped by a cup of coffee; you are supposed to be able to solve the 1-cup exercises.
  - c2) ☕☕ : this is an exercise that requires a cozy spot in a library, silence, and a little more care in the choice of coffee (American coffee proved to be an insufficient adjuvant); you are supposed to be able to solve the 2-cup exercises, with a little help (that *may* be coming from the exercise sessions, wink wink).
  - c3) ☕☕☕ : these are usually the optional exercises. They are supposed to be difficult: don't be put off, enjoy the process of discovery, helped by your favourite psychotropic drug.
- Give the optional exercises a try: a failed attempt based on a good idea will be rewarded.

1. Universal properties, Limits and colimits

> Segundo ejercicio es meditacion de los pecados, y contiene en si, despues de la oracion preparatoria y dos preámbulos, cinco puntos y un coloquio.

☕ Exercise 1

Let $1 = \{\bot\}$ be any set with a single element $\bot$; consider the correspondence that sends a set A into $A \sqcup 1$, i.e. to the coproduct of A and 1.

This is a functor $\_ \sqcup 1$ from the category Set of sets and functions to itself, called the "*add a disjoint point to a set*", or the "*maybe*" construction (for reasons rooted in functional programming).

Show that $\_ \sqcup 1$ is indeed a functor.

Show that the following diagram commutes:

$$\begin{array}{ccc} 1\sqcup 1\sqcup 1 & \xrightarrow{1\sqcup \mathsf{fold}} & 1\sqcup 1 \\ {\scriptstyle \mathsf{fold}\sqcup 1}\downarrow & & \downarrow{\scriptstyle \mathsf{fold}} \\ 1\sqcup 1 & \xrightarrow[\mathsf{fold}]{} & 1 \end{array} \tag{1.1}$$

where $\mathsf{fold} : 1 \sqcup 1 \to 1$ is the unique function from $1 \sqcup 1$ to the point, since 1 is terminal in Set; fold is also obtained from the universal property of the coproduct: in what way?

Show that the universal property of the coproduct yields a unique pair of functions (i_A, i_1), $i_A : A \to A \sqcup 1$, $i_1 : 1 \to A \sqcup 1$.

Show that the set of functions $a : A \sqcup 1 \to A$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{i_A} & A\sqcup 1 \\ & {\scriptstyle id}\searrow\!\!\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} \tag{1.2}$$

commutes is in bijection with the elements of A; this means that each such function $a : A \sqcup 1 \to A$ determines a unique element $a_0 : A$.

☕ Exercise 2

Let $\mathcal{C} = (\mathbb{N}, \geq)$ be the category having objects the natural numbers $0, 1, 2, \ldots$ and where there is a morphism $m \to n$ if and only if $n \leq m$ (read twice this definition). Does $\mathcal{C}$ have a terminal object? Does it have an initial object? Does it have products

$n \times m$? Does it have coproducts $n \sqcup m$? Answer the same four questions for the category $\mathcal{D} = (\mathbb{N}, \_ \mid \_)$, that has the same objects and where there is a morphism $n \to m$ if and only if $m = kn$ for some $k \in \mathbb{N}$ (the relation $n \mid m$ reads as "n divides m").

Exercise 3

Let $\mathcal{C}, \mathcal{D}, \mathcal{Z}$ be three categories, and $\mathcal{C} \xrightarrow{F} \mathcal{Z} \xleftarrow{G} \mathcal{D}$ two functors; define the *comma category* of the functors F, G to be the category (F/G) whose

- objects are arrows in $\mathcal{Z}$ of the form $f : FC \to GD$ for a pair of objects $C \in \mathcal{C}, D \in \mathcal{D}$ (more formally, an object of (F/G) is a tuple $(C, D, f : Fc \to Gd)$);
- morphisms $(C, D, f) \to (C', D', f')$ are pairs $u : C \to C'$, $v : D \to D'$ with the property that the square

$$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ {\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle Gv} \\ FC' & \xrightarrow[f']{} & GD' \end{array} \tag{1.3}$$

  is commutative.
  - Show that (F/G) is indeed a category defining its composition rule and identity maps.
  - What is (F/G) if G is a constant functor (say, at an object Z)? What if G is constant *and* F is the identity functor of $\mathcal{Z}$? What if F, G are *both* constant (say at objects Z, Z')?
  - Show that (F/G) has the following universal property: there exists a pair of functors $\mathcal{C} \times \mathcal{D} \xleftarrow{Q} (F/G) \xrightarrow{P} \mathcal{Z}^{\to}$ (the arrow category of $\mathcal{Z}$) with the property that the square

$$\begin{array}{ccc} (F/G) & \xrightarrow{P} & \mathcal{Z}^{\to} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle \left[\substack{\text{source} \\ \text{target}}\right]} \\ \mathcal{C} \times \mathcal{D} & \xrightarrow[F \times G]{} & \mathcal{Z} \times \mathcal{Z} \end{array} \tag{1.4}$$

    is commutative, and for every other commutative square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{K} & \mathcal{Z}^{\to} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \left[\substack{\text{source} \\ \text{target}}\right]} \\ \mathcal{C} \times \mathcal{D} & \xrightarrow[F \times G]{} & \mathcal{Z} \times \mathcal{Z} \end{array} \tag{1.5}$$

there is a unique functor $\langle H, K\rangle : \mathcal{X} \to (F/G)$ such that $Q \circ \langle H, K\rangle = H$ and $P \circ \langle H, K\rangle = K$. In other words, (F/G) is the pullback of $J = \left[\begin{smallmatrix}\text{source}\\ \text{target}\end{smallmatrix}\right]$ and $F \times G$.

Exercise 4 (*This exercise is optional*)

Let Dyn be the following category:

- Objects are tuples $(X, s; x)$ where $s : X \to X$ is a function on the set X, and $x : X$ is an element;
- Morphisms $(X, s; x) \to (Y, t; y)$ are functions $f : X \to Y$ sending x to y, and such that the diagram

(1.6)
$$\begin{array}{ccc} X & \xrightarrow{s} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[t]{} & Y \end{array}$$

is commutative.

An object of Dyn is called a *dynamical system* because one can study the 'evolution' of s through time, as the family of iterates $\{\mathrm{id}, s, s^2, s^3, \dots\}$ acting on a point $x : X$.

Show that the object $\mathbf{N} = (\mathbb{N}, s, 0)$ is an initial object of Dyn, where

- $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \dots\}$;
- $s : \mathbb{N} \to \mathbb{N} : \lambda n.n + 1$.

What does the universal property of $\mathbf{N}$ mean in terms of an object $(X, s; x)$?

The *mathematical induction principle* says that

(1.7)
$$\big(Q0 \wedge \textstyle\bigwedge_n \bigwedge_{i\le n} Qi \Rightarrow Q(i+1)\big) \Rightarrow \textstyle\bigwedge_{n:\mathbb{N}} Qn$$

(in words, if $Q : \mathbb{N} \to \{0, 1\}$ is a proposition, $Q0$ is true, and $Qn \Rightarrow Q(n + 1)$, then Qn is true for all $n : \mathbb{N}$. Is there a way to state the induction principle in terms of the universal property of $\mathbb{N}$?

Hint: Use the universal property of $\mathbb{N}$ to show that if $S \subseteq \mathbb{N}$ is a nonempty subset such that the inclusion $i : S \hookrightarrow \mathbb{N}$ is a morphism in the category Dyn of discrete dynamical systems, then $S = \mathbb{N}$. Deduce the induction principle using a suitable S_Q obtained from the property Q.