CATEGORY THEORY ITI9200 EXERCISES

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How to approach these exercises

- The ability to think is rewarded more than correctness; this means that good ideas leading to wrong answers are more valuable than bad ideas yielding the correct answer. Category theory is founded on the belief that the right answer is useless when found by means of an unenlightening train of thought.
- Keep in mind that every exercise has just a finite number of correct answers, and an infinite number of wrong answers (wink wink).
- Every exercise is marked with a certain number of ☕'s, according to the Rényi-Erdös complexity scale:
  - c1) ☕ : this is an exercise that merely requires to sit down, think, and solve the puzzle, helped by a cup of coffee; you are supposed to be able to solve the 1-cup exercises.
  - c2) ☕☕ : this is an exercise that requires a cozy spot in a library, silence, and a little more care in the choice of coffee (American coffee proved to be an insufficient adjuvant); you are supposed to be able to solve the 2-cup exercises, with a little help (that *may* be coming from the exercise sessions, wink wink).
  - c3) ☕☕☕ : these are usually the optional exercises. They are supposed to be difficult: don't be put off, enjoy the process of discovery, helped by your favourite psychotropic drug.
- Give the optional exercises a try: a failed attempt based on a good idea will be rewarded.

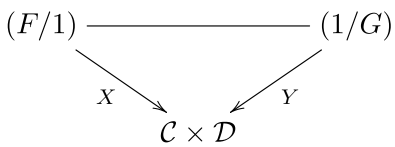
1. Adjunctions, Yoneda, Monads

Tercero ejercicio es repeticion del primero y segundo ejercicio, haciendo tres coloquios.

Exercise 1

Prove that two functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ are adjoints, with F left adjoint to G, if and only if the two comma categories $(F/1_{\mathcal{D}})$ and $(1_{\mathcal{C}}/G)$ are 'equivalent over $\mathcal{C} \times \mathcal{D}$', namely there is an equivalence of categories $U : (F/1_{\mathcal{D}}) \cong (1_{\mathcal{C}}/G) : V$ with the property that the diagram

(1.1)



is commutative (choosing either U or its inverse V as horizontal arrow).

Here, $X : (F/1) \to \mathcal{C} \times \mathcal{D}$ is the functor that sends an object $(C, D, FC \to D)$ to the pair (C, D), and similarly Y sends an object $(C, D, C \to GD)$ to the pair (C, D).

Answer of exercise 1

Let's say that F is left adjoint to G. A correspondence on objects is easily defined sending an object $f : FC \to D$ into its corresponding object

$$C \xrightarrow{\eta_C} GFC \xrightarrow{Gf} GD \tag{1.2}$$

in $(1_{\mathcal{C}}/G)$, using the unit of the adjunction; the inverse of this correspondence uses the counit of the adjunction, sending a $g : C \to GD$ into

$$FC \xrightarrow{Fg} FGD \xrightarrow{\epsilon_D} D. \tag{1.3}$$

The adjunction identities entail that these two correspondences are mutually inverse. All in all, we have just observed that the bijections

$$\mathcal{D}(FC, D) \cong \mathcal{C}(C, GD) \tag{1.4}$$

that the adjunction determines attach to a bijection $\coprod_{C,D} \mathcal{D}(FC, D) \cong \coprod_{C,D} \mathcal{C}(C, GD)$ between the classes of objects of $(F/1)$ and $(1/G)$.

The fact that the above correspondence, exploiting the adjunction identities, lifts to a pair of mutually inverse functors boils down to the fact that there is a bijection between commutative squares on the left and on the right of

$$
\left\{ \begin{array}{ccc} FC & \xrightarrow{f} & D \\ {\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle v} \\ FC' & \xrightarrow[f']{} & D' \end{array} \right\} \cong \left\{ \begin{array}{ccc} C & \xrightarrow{g} & GD \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle Gv} \\ C' & \xrightarrow[g']{} & GD' \end{array} \right\} \tag{1.5}
$$

It is easily seen that the naturality of η, ϵ, as well as the mentioned adjunction identities, realise this bijection.

Exercise 2 I am Yo(ne)da lemma, Luke

Show that the contravariant Yoneda lemma is equivalent to the following statement:

Given a small category $\mathcal{C}$, an object $A \in \mathcal{C}$, and a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, there is an equaliser diagram

$$
FA \longrightarrow \prod_{U \in \mathcal{C}} \mathsf{Set}(\mathcal{C}(U, A), FU) \underset{r}{\overset{l}{\rightrightarrows}} \prod_{f : X \to Y} \mathsf{Set}(\mathcal{C}(Y, A), FX) \tag{$\star$}
$$

where the left product is indexed over the objects of $\mathcal{C}$, the right product is indexed over the morphisms of $\mathcal{C}$, and the functions l, r are respectively defined as follows. Given an element

$$\Theta_\bullet = \{\Theta_U : \mathcal{C}(U, A) \to FU \mid U \in \mathcal{C}_o\}$$

in the product $\prod_{U \in \mathcal{C}} \mathsf{Set}(\mathcal{C}(U, A), FU)$, we define

- the composite map

  $$\prod_{U \in \mathcal{C}} \mathsf{Set}(\mathcal{C}(U, A), FU) \xrightarrow{l} \prod_{f : X \to Y} \mathsf{Set}(\mathcal{C}(Y, A), FX) \xrightarrow{\pi_f} \mathsf{Set}(\mathcal{C}(Y, A), FX)$$

  sending

  $$\Theta_\bullet \mapsto \Big(\mathcal{C}(Y, A) \xrightarrow{\Theta_Y} FY \xrightarrow{Ff} FX\Big);$$

- the composite map

  $$\prod_{U \in \mathcal{C}} \mathsf{Set}(\mathcal{C}(U, A), FU) \xrightarrow{r} \prod_{f : X \to Y} \mathsf{Set}(\mathcal{C}(Y, A), FX) \xrightarrow{\pi_f} \mathsf{Set}(\mathcal{C}(Y, A), FX)$$

  sending

  $$\Theta_\bullet \mapsto \Big(\mathcal{C}(Y, A) \xrightarrow{\_\circ f} \mathcal{C}(X, A) \xrightarrow{\Theta_X} FX\Big).$$

If each $\pi_f : \prod_{f : X \to Y} \mathsf{Set}(\mathcal{C}(Y, A), FX) \to \mathsf{Set}(\mathcal{C}(Y, A), FX)$ is the projection from the product, at the component indexed by $f : X \to Y$, the totality of the compositions $\pi_f \circ l, \pi_f \circ r$ uniquely determines l, r.

Optional: show that the equaliser in ($\star$) is natural in the object A.

Answer of exercise 2

Exploiting the definition of the maps l, r, one sees that an element in the equaliser is a family $\Theta_\bullet$ with the property that for every $f : X \to Y$ the square

$$\begin{array}{ccc} \mathcal{C}(Y,A) & \xrightarrow{\Theta_Y} & FY \\ {\scriptstyle \mathcal{C}(f,A)}\big\downarrow & & \big\downarrow{\scriptstyle Ff} \\ \mathcal{C}(X,A) & \xrightarrow[\Theta_X]{} & FX \end{array}$$

is commutative; evidently, this is just the request that the family above defines a natural transformation $\Theta : \mathcal{C}(-,A) \Rightarrow F$. Thus, the equaliser of the pair (l,r) is just the set of such natural transformations, regarded as a subset of $\prod_{U\in\mathcal{C}} \mathsf{Set}(\mathcal{C}(U,A), FU)$.

Thus, in order to solve the exercise, it is enough to show that FA has the same universal property:

- there is a map $z : FA \to \prod_{U\in\mathcal{C}} \mathsf{Set}(\mathcal{C}(U,A), FU)$ equalising (l,r);
- for every other map $h : E \to \prod_{U\in\mathcal{C}} \mathsf{Set}(\mathcal{C}(U,A), FU)$ equalising (l,r), there is a unique $E \to FA$ closing the triangle in

$$\begin{array}{ccccc} FA & \xrightarrow{z} & \prod_{U\in\mathcal{C}} \mathsf{Set}(\mathcal{C}(U,A), FU) & \overset{l}{\underset{r}{\rightrightarrows}} & \prod_{f:X\to Y} \mathsf{Set}(\mathcal{C}(Y,A), FX) \\ {\scriptstyle \bar{h}}\,\big\uparrow & \nearrow_{h} & & & \\ E & & & & \end{array}$$

Let's first define z: an element $a : FA$ goes to the family of functions

$$\{\lambda u.Fu(a) \mid u : U \to A, U \in \mathcal{C}\}$$

regarded as an element of $\prod_{U\in\mathcal{C}} \mathsf{Set}(\mathcal{C}(U,A), FU)$. Clearly, z equalises (l,r): it does if and only if it equalises the pair (l_f, r_f) for every $f : X \to Y$, and

$$\lambda u.l_f(z(a)) = \lambda u.Ff(z(a)_u) = \lambda u.(Ff(Fu(a)))$$
$$\lambda u.r_f(z(a)) = \lambda u.z(a)(f^* \circ u) = \lambda u.F(uf)(a)$$

are clearly equal because F is a (contravariant) functor.

Now, given another cone

$$h : E \to \prod_{U\in\mathcal{C}} \mathsf{Set}(\mathcal{C}(U,A), FU)$$

by the universal property of the product at its codomain, h has components

$$h_U : E \to \mathsf{Set}(\mathcal{C}(U,A), FU)$$

one for each $U \in \mathcal{C}$. The fact that h equalises (l, r) this means that the two compositions

$$h_U(e) : \mathcal{C}(U, A) \longrightarrow FU \rightrightarrows \begin{cases} \big(\mathcal{C}(Y, A) \xrightarrow{h_Y(e)} FY \xrightarrow{Ff} FX\big) \\ \big(\mathcal{C}(Y, A) \xrightarrow{f^*} \mathcal{C}(X, A) \xrightarrow{h_X(e)} FX\big) \end{cases}$$

are equal. But then, $h(e)$ is completely determined by its component at the identity:

$$h(e)u = Fu.h(e)(\mathrm{id}_A),$$

and the element $h(e)(\mathrm{id}_A)$ lies in FA by construction. The assignment

$$\bar{h} : E \to FA : e \mapsto h(e)(\mathrm{id}_A)$$

thus completely determines h.

Exercise 3 Chains of adjoints, an illustrated guide

Let $(\mathbb{Z}, \leq)$ be the totally ordered set of integers, regarded as a category, and $f : \mathbb{Z} \to \mathbb{Z}$ a monotone function, regarded as an endofunctor. Show that the following conditions are equivalent:

C1) f has a left adjoint f_L;
C2) f has a right adjoint f_R;
C3) the image $f(\mathbb{Z})$ of f is unbounded from below and from above.

(hint: show that f has a right adjoint if and only if the following condition holds:

D1) each set $S_m = \{n \mid fn \leq m\}$ is nonempty and bounded from above; thus $f_R(m) := \max S_m$.

Show that this, in turn, is equivalent to the third condition above. Dualise for left adjoints.)

Let $f : \mathbb{Z} \to \mathbb{Z}$ be the map sending an integer k to $2k$, so that the image of f is $2\mathbb{Z}$; what are the left (and the right) adjoints f_L, f_R of f?

Describe the monads obtained from the adjunction $f_L \dashv f$ and from the adjunction $f \dashv f_R$.

Answer of exercise 3

Let's show that C2 is equivalent to D1:

- If f has a right adjoint f_R, then

$$fn \leq m \iff n \leq f_R m \tag{1.6}$$

so that S_m is nonempty (it has as many elements as there are ns such that $n \leq f_R m$, and this set is nonempty), and bounded from above (by $f_R m$).

- Conversely, if each S_m is nonempty and bounded from above, $f_R m := \max S_m$ defines a right adjoint for f, because

$$fn \le m \iff n \in S_m \iff n \le \max S_m. \tag{1.7}$$

Now, let's show that D1 is equivalent to C3:

- Assume D1 is not true; then there exists an empty S_m (and in that case $fn \ge m$ for every $n \in \mathbb{Z}$) or an S_m unbounded from above (and in that case, there exists no maximum element in S_m, so that for every n such tat $fn \le m$, there exists a $n' > n$ such that $fn' \le m$).
- With a similar chain of reasoning, one proves that if c3 is not true, then d1 is not true.

In a similar fashion, one shows the claim for a left adjoint to f.

In case f is the 'multiply by two' maps, given an $n \in \mathbb{Z}$, the set $S_m = \{n \mid 2n \le m\}$ has $\lceil \frac{n}{2} \rceil$ as maximum element. This defines the right adjoint f_R; it is evident what is the action of the monad ff_R.

☕☕☕ Exercise 4 Every functor is secretly a monad (*This exercise is optional*)

Let

$$\mathcal{B} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

be two functors with common domain. For the purposes of this exercise, a *right extension* of F along G is a pair $\langle R_G F, \epsilon \rangle$ where $R_G F : \mathcal{B} \to \mathcal{C}$ is a functor, and $\epsilon : R_G(F) \circ G \Rightarrow F$ a natural transformation with the following universal property:

> Composition with $\epsilon : R_G(F) \circ G \Rightarrow F$ induces a bijection
>
> $$\frac{H \Rightarrow R_G F}{HG \Rightarrow F}$$

More explicitly, the bijection is induced by $\big(\alpha : H \Rightarrow R_G F\big) \mapsto \big(H \circ G \Rightarrow R_G F \circ G \overset{\epsilon}{\Rightarrow} F\big)$.

Define a functor $M(F) : \mathcal{D} \to \mathcal{D}$ to be the right Kan extension of F along itself.

Show that it is a monad on $\mathcal{D}$ (find multiplication and unit using the universal property, show the monad axioms). This is called the *codensity monad* of F.

Show that when F has a left adjoint L then $M(F) \cong FL$ (the monad generated by the adjunction $L \dashv F$).

Find $M(F)$ in the following cases (if they exist; if they don't, show why):

- when $F : \mathbb{Z} \to \mathbb{R}$ is the inclusion of abelian groups;
- when $F : \mathbb{Z} \to \mathbb{R}$ is the inclusion of totally ordered sets;
- when $F : \{0 \to 1\} \to \{a \cong b\}$ sends the non-identity arrow $0 \to 1$ to the isomorphism $a \to b$ in the codomain.