CATEGORY THEORY ITI9200 EXERCISES

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How to approach these exercises

- The ability to think is rewarded more than correctness; this means that good ideas leading to wrong answers are more valuable than bad ideas yielding the correct answer. Category theory is founded on the belief that the right answer is useless when found by means of an unenlightening train of thought.
- Keep in mind that every exercise has just a finite number of correct answers, and an infinite number of wrong answers (wink wink).
- Every exercise is marked with a certain number of ☕'s, according to the Rényi-Erdös complexity scale:
  - c1) ☕ : this is an exercise that merely requires to sit down, think, and solve the puzzle, helped by a cup of coffee; you are supposed to be able to solve the 1-cup exercises.
  - c2) ☕☕ : this is an exercise that requires a cozy spot in a library, silence, and a little more care in the choice of coffee (American coffee proved to be an insufficient adjuvant); you are supposed to be able to solve the 2-cup exercises, with a little help (that *may* be coming from the exercise sessions, wink wink).
  - c3) ☕☕☕ : these are usually the optional exercises. They are supposed to be difficult: don't be put off, enjoy the process of discovery, helped by your favourite psychotropic drug.
- Give the optional exercises a try: a failed attempt based on a good idea will be rewarded.

1. Adjunctions, Yoneda, Monads

> Tercero ejercicio es repeticion del primero y segundo ejercicio, haciendo tres coloquios.

Exercise 1

Prove that two functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ are adjoints, with F left adjoint to G, if and only if the two comma categories $(F/1_\mathcal{D})$ and $(1_\mathcal{C}/G)$ are 'equivalent over $\mathcal{C} \times \mathcal{D}$', namely there is an equivalence of categories $U : (F/1_\mathcal{D}) \cong (1_\mathcal{C}/G) : V$ with the property that the diagram

$$\begin{array}{ccc} (F/1) & \text{———} & (1/G) \\ & {\scriptstyle X}\searrow \quad \swarrow{\scriptstyle Y} & \\ & \mathcal{C} \times \mathcal{D} & \end{array} \tag{1.1}$$

is commutative (choosing either U or its inverse V as horizontal arrow).

Here, $X : (F/1) \to \mathcal{C} \times \mathcal{D}$ is the functor that sends an object $(C, D, FC \to D)$ to the pair (C, D), and similarly Y sends an object $(C, D, C \to GD)$ to the pair (C, D).

Exercise 2 I am Yo(ne)da lemma, Luke

Show that the contravariant Yoneda lemma is equivalent to the following statement:

Given a small category $\mathcal{C}$, an object $A \in \mathcal{C}$, and a functor $F : \mathcal{C}^{\text{op}} \to \mathsf{Set}$, there is an equaliser diagram

$$FA \longrightarrow \prod_{U \in \mathcal{C}} \mathsf{Set}(\mathcal{C}(U, A), FU) \underset{r}{\overset{l}{\rightrightarrows}} \prod_{f : X \to Y} \mathsf{Set}(\mathcal{C}(Y, A), FX) \tag{$\star$}$$

where the left product is indexed over the objects of $\mathcal{C}$, the right product is indexed over the morphisms of $\mathcal{C}$, and the functions l, r are respectively defined as follows. Given an element

$$\Theta_\bullet = \{\Theta_U : \mathcal{C}(U, A) \to FU \mid U \in \mathcal{C}_o\}$$

in the product $\prod_{U \in \mathcal{C}} \mathsf{Set}(\mathcal{C}(U, A), FU)$, we define

- the composite map

$$\prod_{U \in \mathcal{C}} \mathsf{Set}(\mathcal{C}(U, A), FU) \xrightarrow{l} \prod_{f : X \to Y} \mathsf{Set}(\mathcal{C}(Y, A), FX) \xrightarrow{\pi_f} \mathsf{Set}(\mathcal{C}(Y, A), FX)$$

sending

$$\Theta_\bullet \mapsto \Big(\mathcal{C}(Y,A) \xrightarrow{\Theta_Y} FY \xrightarrow{Ff} FX\Big);$$

- the composite map

$$\prod_{U\in\mathcal{C}} \mathsf{Set}(\mathcal{C}(U,A), FU) \xrightarrow{r} \prod_{f:X\to Y} \mathsf{Set}(\mathcal{C}(Y,A), FX) \xrightarrow{\pi_f} \mathsf{Set}(\mathcal{C}(Y,A), FX)$$

sending

$$\Theta_\bullet \mapsto \Big(\mathcal{C}(Y,A) \xrightarrow{\_\circ f} \mathcal{C}(X,A) \xrightarrow{\Theta_X} FX\Big).$$

If each $\pi_f : \prod_{f:X\to Y} \mathsf{Set}(\mathcal{C}(Y,A), FX) \to \mathsf{Set}(\mathcal{C}(Y,A), FX)$ is the projection from the product, at the component indexed by $f : X \to Y$, the totality of the compositions $\pi_f \circ l, \pi_f \circ r$ uniquely determines l, r.

Optional: show that the equaliser in ($\star$) is natural in the object A.

☕☕ Exercise 3 Chains of adjoints, an illustrated guide

Let $(\mathbb{Z}, \leq)$ be the totally ordered set of integers, regarded as a category, and $f : \mathbb{Z} \to \mathbb{Z}$ a monotone function, regarded as an endofunctor. Show that the following conditions are equivalent:

C1) f has a left adjoint f_L;
C2) f has a right adjoint f_R;
C3) the image $f(\mathbb{Z})$ of f is unbounded from below and from above.

(hint: show that f has a right adjoint if and only if the following condition holds:

D1) each set $S_m = \{n \mid fn \leq m\}$ is nonempty and bounded from above; thus $f_R(m) := \max S_m$.

Show that this, in turn, is equivalent to the third condition above. Dualise for left adjoints.)

Let $f : \mathbb{Z} \to \mathbb{Z}$ be the map sending an integer k to $2k$, so that the image of f is $2\mathbb{Z}$; what are the left (and the right) adjoints f_L, f_R of f?

Describe the monads obtained from the adjunction $f_L \dashv f$ and from the adjunction $f \dashv f_R$.

☕☕☕ Exercise 4 Every functor is secretly a monad (*This exercise is optional*)

Let

$$\mathcal{B} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

be two functors with common domain. For the purposes of this exercise, a *right extension* of F along G is a pair $\langle R_G F, \epsilon\rangle$ where $R_G F : \mathcal{B} \to \mathcal{C}$ is a functor, and $\epsilon : R_G(F) \circ G \Rightarrow F$ a natural transformation with the following universal property:

Composition with $\epsilon : R_G(F) \circ G \Rightarrow F$ induces a bijection

$$\frac{H \Rightarrow R_G F}{HG \Rightarrow F}$$

More explicitly, the bijection is induced by $\big(\alpha : H \Rightarrow R_G F\big) \mapsto \big(H \circ G \Rightarrow R_G F \circ G \stackrel{\epsilon}{\Rightarrow} F\big)$.

Define a functor $M(F) : \mathcal{D} \to \mathcal{D}$ to be the right Kan extension of F along itself.

Show that it is a monad on $\mathcal{D}$ (find multiplication and unit using the universal property, show the monad axioms). This is called the *codensity monad* of F.

Show that when F has a left adjoint L then $M(F) \cong FL$ (the monad generated by the adjunction $L \dashv F$).

Find $M(F)$ in the following cases (if they exist; if they don't, show why):

- when $F : \mathbb{Z} \to \mathbb{R}$ is the inclusion of abelian groups;
- when $F : \mathbb{Z} \to \mathbb{R}$ is the inclusion of totally ordered sets;
- when $F : \{0 \to 1\} \to \{a \cong b\}$ sends the non-identity arrow $0 \to 1$ to the isomorphism $a \to b$ in the codomain.