CATEGORY THEORY ITI9200 EXERCISES

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How to approach these exercises

- The ability to think is rewarded more than correctness; this means that good ideas leading to wrong answers are more valuable than bad ideas yielding the correct answer. Category theory is founded on the belief that the right answer is useless when found by means of an unenlightening train of thought.
- Keep in mind that every exercise has just a finite number of correct answers, and an infinite number of wrong answers (wink wink).
- Every exercise is marked with a certain number of ☕'s, according to the Rényi-Erdös complexity scale:
  - c1) ☕ : this is an exercise that merely requires to sit down, think, and solve the puzzle, helped by a cup of coffee; you are supposed to be able to solve the 1-cup exercises.
  - c2) ☕☕ : this is an exercise that requires a cozy spot in a library, silence, and a little more care in the choice of coffee (American coffee proved to be an insufficient adjuvant); you are supposed to be able to solve the 2-cup exercises, with a little help (that *may* be coming from the exercise sessions, wink wink).
  - c3) ☕☕☕ : these are usually the optional exercises. They are supposed to be difficult: don't be put off, enjoy the process of discovery, helped by your favourite psychotropic drug.
- Give the optional exercises a try: a failed attempt based on a good idea will be rewarded.

1. Miscellaneous exercises

Cuarto ejercicio es resumiendo este mismo tercero.

Quinto ejercicio es meditacion del infierno.

Exercise 1

Let Set_* be the category of *pointed sets*;

- Show that the category Set_* of pointed sets and point-preserving functions $f : (A, a) \to (B, b)$ such that $f(a) = b$ is just the "coslice" category $*/\mathsf{Set}$ of arrows $a : * \to A$ and commutative triangles, where $* = \{\bullet\}$ is a singleton set;
- given two pointed sets $x : * \to X$ and $y : * \to Y$, define the *wedge sum* $X \vee Y$ of $(X, x), (Y, y)$ as the result of the following pushout:

$$\begin{array}{ccc} * & \xrightarrow{x} & X \\ {\scriptstyle y}\downarrow & & \downarrow \\ Y & \longrightarrow & X \vee Y \end{array} \tag{1.1}$$

  Show that $\_ \vee \_ : \mathsf{Set}_* \times \mathsf{Set}_* \to \mathsf{Set}_*$ is a bifunctor, and show that it is part of a symmetric monoidal structure on Set_*.

Exercise 2

Let $\mathcal{C}$ be a 2-category; a *right extension* of f along g in $\mathcal{C}$ consists of a triangle

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle g}\swarrow & \overset{\epsilon}{\Rightarrow} & \searrow{\scriptstyle f} & \\ Y & & \xrightarrow[r_{g,f}]{} & & Z \end{array} \tag{1.2}$$

filled by a 2-cell $\epsilon : r_{g,f} \circ g \Rightarrow f$ with the property that for every other 2-cell

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle g}\swarrow & \overset{\zeta}{\Rightarrow} & \searrow{\scriptstyle f} & \\ Y & & \xrightarrow[k]{} & & Z \end{array} \tag{1.3}$$

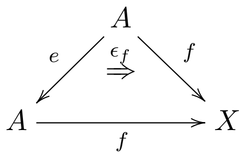
there exists a unique $\bar\zeta : k \Rightarrow r_{g,f}$ with the property that the composition $k \circ g \overset{\bar\zeta * g}{\Rightarrow} r_{g,f} \circ g \overset{\epsilon}{\Rightarrow} f$ equals the 2-cell $\zeta : k \circ g \Rightarrow f$.

One often says that the pair $(\epsilon, r_{g,f})$ *exhibits* the right extension of f along g.

Now, let $e : A \to A$ be a 1-cell of an object A into itself; show that the following conditions are equivalent:

- e is isomorphic to the identity 1-cell $1_A : A \to A$ of the object A;
- for every $f : A \to X$, the triangle

(1.4)



is the right extension of f along e for a certain ϵ_f which is an isomorphism. In other words, (ϵ, f) exhibits the right extension of f along e, and ϵ is invertible.

(Hint: choose $X = A$ and $f = \dots$; now, since ϵ_f is an isomorphism...)

Exercise 3

A *simple polynomial* is a functor $F : \mathsf{Set} \to \mathsf{Set}$ that is defined from the following inductive rules:

SP1) the identity functor $X \mapsto X$ is a simple polynomial;
SP2) every constant functor $X \mapsto A$ is a simple polynomial;
SP3) the product $F \times G : X \mapsto FX \times GX$ of two simple polynomials is simple;
SP4) the coproduct $\coprod_{i\in I} F_i : X \mapsto \coprod_{i \in I} F_i X$ of an arbitrary number of simple polynomials is simple.

An example of a polynomial functor is $X \mapsto A \times X^3 + B \times X^2 + X + 1$, where $\times$ denotes cartesian product, and $+$ denotes coproduct; another example is a 'formal series functor' $X \mapsto \coprod_{i\in I} A_i \times X^{n_i}$ where n_i are natural numbers and $(A_i \mid i \in I)$ is an arbitrary family of sets.

An *arity function* consists of a set I equipped with a function $a : I \to \mathbb{N}$; the inverse image $a^{-1}n$ is the set of elements in I having 'arity' n.[1] Every arity function $a : I \to \mathbb{N}$ defines an *arity functor* as

$$F_a : X \mapsto \coprod_{i\in I} X^{a(i)} = \{(i, \underline{x}) \mid i \in I,\, \underline{x} \in X^{a(i)}\} \tag{1.5}$$

Show that the class of simple polynomials coincides with the class of arity functors (F_a is 'clearly' a simple polynomial: how does one define an arity associated to a given simple polynomial?)

[1] The word 'arity' is a back-formation from the Latin adjectival numeral suffix *-arius*, used to form adjectives from nouns or numerals.

Exercise 4

A notable result in the theory of adjoint functors is the *adjoint functor theorem*, establishing ufficient condition for the existence of a left adjoint to a functor $F : \mathcal{C} \to \mathcal{D}$ that preserves all limits: a functor F that preserves all limits has a left adjoint if and only if it satisfies a certain condition known as the *solution set condition*; a key result towards the proof of this equivalence is the following *initial object lemma*:

> Let $\mathcal{C}$ be a category admitting all small limits; then, $\mathcal{C}$ has an initial object if and only if it has a *weakly initial family*, i.e. a set of objects $\{W_i \mid i \in I\}$ with the property that for every $X \in \mathcal{C}$ there exists at least (but possibly many) arrow $W_{i(X)} \to X$.

Prove the initial object lemma, following this guide:

- If $\mathcal{C}$ has an initial object, it obviously has a weakly initial family;
- Conversely, build the product $W = \prod_{i \in I} W_i$ of all the elements of a weakly initial family.
- Consider the joint equaliser

$$K \xrightarrow{k} W \underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}} W \tag{1.6}$$

  of all endomorphisms of W (this means that k has the property that $ku = kv$ for every pair $u, v : W \to W$, and it is terminal with this property);
- K is a weakly initial object: why? Show that K is an initial object: assume $f, g : K \to X$ are parallel arrows out of K; show that $f = g$ (hint: $f = g$ if and only if their equaliser is isomorphic to K).

Exercise (*This exercise is optional*)

This exercise is intended to show that proving simple statements about monoidal categories requires a lot of work.

> Show that the monoidal unit in a monoidal category $(\mathcal{C}, \cdot)$ is unique up to a unique isomorphism, in the sense that if I, J are two objects playing the rôle of monoidal unit, then there exists a *unique* isomorphism $u : I \cong J$.

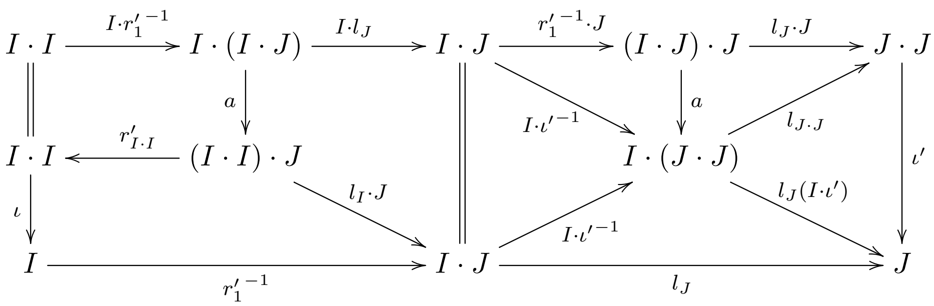
Start by showing that you can define an isomorphism $\eta : I \to J$ by composition

$$I \longrightarrow I \cdot J \longrightarrow J \tag{1.7}$$

Now, show that the square

(1.8)
$$\begin{array}{ccc} I \cdot I & \xrightarrow{\eta \cdot \eta} & J \cdot J \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota'} \\ I & \xrightarrow[\eta]{} & J \end{array}$$

where $\iota = r_1 = l_1$, $\iota' = r'_{1'} = l'_{1'}$, is commutative, splitting the diagram as follows:

(1.9)


(these commutativities are many, and some might require additional lemmas).

From this, show that the object I has the following property: if

(1.10)
$$\begin{array}{ccc} I \cdot I & \xrightarrow{u \cdot u} & I \cdot I \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota'} \\ I & \xrightarrow[u]{} & I \end{array}$$

is a commutative diagram, and u is an isomorphism, then u is the identity arrow of the object I. Conclude that η is the unique isomorphism $I \to J$.

☕☕☕ Exercise (*This exercise is optional*)

Recall the definition of the category Dyn of (unpointed) *dynamical systems*:

- Objects are pairs (X, s) where $s : X \to X$ is a function on the set X;
- Morphisms $(X, s) \to (Y, t)$ are functions $f : X \to Y$ such that the diagram

(1.11)
$$\begin{array}{ccc} X & \xrightarrow{s} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[t]{} & Y \end{array}$$

is commutative.

A dynamical system (X, s) is called *reversible* if $s : X \to X$ is an invertible function. A morphism between two reversible dynamical systems is just a morphism of dynamical systems.

Show that the inclusion functor $\mathsf{RevDyn} \hookrightarrow \mathsf{Dyn}$ admits a left adjoint, i.e. that for every morphism

$$f : (X, s) \longrightarrow (A, \sigma) \tag{1.12}$$

where (A, σ) is a reversible dynamical system, there exist

- A reversible dynamical system $(\bar{X}, \bar{s})$ with a map $u : (X, s) \to (\bar{X}, \bar{s})$ of dynamical systems;
- a *unique* $\bar{f} : \bar{X} \to A$ which is a morphism of dynamical systems, with the property that $\bar{f} \circ u = f$:

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle u}\downarrow & \nearrow_{\bar{f}} & \\ \bar{X} & & \end{array} \tag{1.13}$$

  thus realising the isomorphism

$$\mathsf{Dyn}(X, A) \cong \mathsf{RevDyn}(\bar{X}, A). \tag{1.14}$$