Category theory and its applications - ITI9200

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Questions ??? ✓

1)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

$A \circlearrowleft id_A$

$\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$

The relation "being isomorphic" is an eq. rel. on $\mathcal{C}_0$

I) $f : A \longrightarrow B$ is an isomorphism $g : B \longrightarrow A$, g "inverse" of f

$f \cdot g = 1_B$ & $g \cdot f = 1_A$

II) EQ. REL on $\mathcal{C}_0$ ($\sim$)

1. $A \sim A$ refl
2. $\boxed{A \sim B} \rightarrow B \sim A$ sym
3. $A \sim B, B \sim C \rightarrow A \sim C$

1 $A \longrightarrow A$ identity ✓

2 $A \xrightarrow{\boxed{f}} B$, $B \xrightarrow{\boxed{g}} A$, if g is the inverse of f ...

then f is the inverse of g !

Remark If $f : A \to B$ is an isomorphism, then its inverse is unique.

Proof assume that g, g' are both inverses $B \to A$ of $f : A \to B$.

$$(*)\ \begin{cases} fg = 1_B \\ gf = 1_A \end{cases} \qquad \begin{cases} fg' = 1_B \ (**) \\ g'f = 1_A \end{cases} \qquad \overset{?}{\dashrightarrow} \qquad g = g'$$

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ B \arrow[u, "g"] \arrow[ur, "1_B"'] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] & A \\ A \arrow[u, "f"] \arrow[ur, "1_A"'] \end{tikzcd}$$

(same for g')

$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \\ \bullet \arrow[u, "g'"] \arrow[ur] \end{tikzcd}$$

glue them together

$$\boxed{\begin{tikzcd} & \bullet \arrow[r, "g"] & \bullet \\ \bullet \arrow[ur, "1_B"] \arrow[r] & \bullet \arrow[u, "f"] \arrow[ur, "1_A"'] \end{tikzcd}}$$

$$g' = 1 \cdot g' \overset{(*)}{=} (g \cdot f) \cdot g' = g \cdot f \cdot g' = g \cdot (fg') = g \cdot 1_B = g$$

$$\boxed{g'} = 1\cdot g' \overset{*}{=} \underline{(g\cdot f)}\cdot g' = \underline{g\cdot f\cdot g'} = g\cdot\underline{(f\cdot g')}^{**} = g\cdot 1 = \boxed{g}$$

QED.

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This small lemma ($f : A \to B$) allows to speak of the inverse of a given f, and write $g = f^{-1}$; $\begin{cases} f\cdot f^{-1} = 1_B \\ f^{-1}\cdot f = 1_A \end{cases}$

↳ If $A \sim B$ via $f : A \to B$ with inverse f^{-1} then $B \sim A$ via $f^{-1} : B \to A$ with inverse $(f^{-1})^{-1} = f$

Being isomorphic is transitive.

$\underbrace{\text{If } \underline{A \sim B} \;\&\; \underline{B \sim C}}$, then $\underline{\underline{A \sim C}}$ ←

$$\begin{cases} f : A \to B, \\ g : B \to C, \end{cases} \quad \boxed{\begin{matrix} f^{-1} : B \to A \\ g^{-1} : C \to B \end{matrix}} \quad \begin{matrix} (\ldots) \\ (\ldots) \end{matrix}$$

How to build an isomorphism $A \xrightarrow{h} C =$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[rr, bend left, "g\cdot f"] & B \arrow[l, shift left, "f^{-1}"] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "g^{-1}"] \end{tikzcd}$$

?) Is $\underline{h = gf}$ an isomorphism

Is the composition of isoms still an isom?

- To build an inverse for $h: A \to C$, compose $C \xrightarrow{g^{-1}} B \xrightarrow{f^{-1}} A$

$$\underline{f^{-1} \cdot g^{-1}}$$

$$g (\underline{f \cdot f^{-1}}) g^{-1} = (g \cdot 1) \cdot g^{-1} = \underline{g \cdot g^{-1}} = 1$$

$$f^{-1} \cdot \underline{g^{-1} \cdot g} \cdot f = f^{-1} \cdot 1 \cdot f = f^{-1} \cdot f = 1 \quad \blacksquare$$

↳ "Being isomorphic" is <u>transitive</u>.

<u>symm.</u>

<u>reflexive</u>

$\Rightarrow$ Eq. relation.

2) $\mathcal{C} = (\underset{\text{obj.}}{\mathcal{C}_0}, \underset{\text{mor}}{\mathcal{C}_1})$. Prove that "there exists an arrow" is a preorder relation on $\mathcal{C}_0$.

"$A \leq B$" if there exists an arrow $f : A \longrightarrow B$

$\mathcal{C}_0$ set, a preorder "$\leq$" is a relation
1) reflexive $A \leq A$
2) transitive $\dfrac{A \leq B, B \leq C}{A \leq C.}$

1) $A \leq A$: True! Because, by axiom, there always exists at least $1_A : A \longrightarrow A$.

2) $\left.\begin{array}{l} A \leq B : \exists\ u : A \longrightarrow B \\ B \leq C : \exists\ v : B \longrightarrow C \end{array}\right\}$ compose them!

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dr, "v \cdot u"'] & B \arrow[d, "v"] \\ & C \end{tikzcd}$$

Since there exists a composition operation on $\mathcal{C}$, given arrows $\underset{A \leq B}{A \longrightarrow B}$, $\underset{B \leq C}{B \longrightarrow C}$, their composition witnesses that $A \leq C$

$A \longrightarrow C$

Recall the def. of PARTIALLY ORDERED set $(X, \leq)$ refl. trans. & antisymm.

$(A, B \in X \quad A \leq B\ \&\ B \leq A)$
$\Rightarrow A = B$

$\begin{array}{l} A \leq B \\ B \leq A \end{array} \longrightarrow A = B$

In categories, "$\exists$ a morphism" is <u>very</u> rarely antisymmetric.

Antisymm. says that 1) $A \le B$
2) $B \le A$ $\longrightarrow$ $A = B$

$A = \{x\ y\ z\} \longrightarrow \{x\ y\ z\}$

1) $\exists\ u : A \longrightarrow B$
2) $\exists\ v : B \longrightarrow A$

$\underline{vu} : A \longrightarrow A$ can be anything in $\hom(A,A)$
$\underline{uv} : B \longrightarrow B$ " " " in $\hom(B,B)$.

(preset)
In a preorder(ed set)

$$\begin{tikzcd}
 & X_1 \arrow[r, "u_2"] & X_2 \arrow[dr, "u_3"] & \\
A \arrow[ur, "u_1"] & & & X_3 \arrow[dl, "u_4"] \\
 & X_5 \arrow[ul, "u_6"] & X_4 \arrow[l, "u_5"] &
\end{tikzcd}$$

This composition can be <u>any</u> element of $\hom(A,A)$ ($u_6 u_5 \ldots u_1$ form a "<u>loop</u>")

$$\begin{matrix} A & \le & X_1 & \le & X_2 \\ \rotatebox{90}{$\le$} & & & & \rotatebox{90}{$\ge$} \\ X_5 & & \ldots & & \end{matrix}$$

in a preorder loops are allowed (& in categories)

OTOH in posets <u>antisymmetry</u> prevents loops from existing $(A \le B \le A \Rightarrow A = B)$

$A \underset{v}{\overset{u}{\rightleftarrows}} B$ is a loop ; in a category $\mathcal{C}$ there is no reason for this loop to be trivial

MISTAKE
WRONG " Since $\exists\ 1_A : A \longrightarrow A$, 1_A is the <u>only</u> morphism from $A \longrightarrow A$." $\leftarrow$ FALSE !

(Def (Thin category) $\uparrow$ this is true)

A category is called **thin** if given two objects A, B, there exists <u>at most one</u> morphism $A \longrightarrow B$.

- So, in a thin category $\hom(A,B)$ is either empty or a single element

★ • Every preordered set $(P, \leq)$ can be regarded as a thin category

- Let $\mathcal{C}$ be a category; let $(\mathcal{C}_0, \leq)$ the associated preordered set

"there exists a morph"

★ entails that every category $\mathcal{C}$ possesses an associated thin category obtained as $(\mathcal{C}_0, \leq)$

3) Every set A, defines a "minimal" and a "maximal" cat having A as set of objects

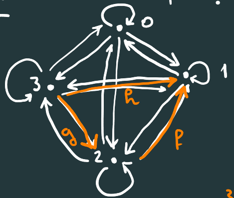
$m(A)_0 = A$

morphisms $a \longrightarrow a' = \begin{cases} \{1_a\} & \text{if } a = a' \\ \emptyset \text{ empty set} & \text{if } a \neq a' \end{cases}$

IT IS A CATEGORY bc. all axioms hold (trivially)

$M(A)_0 = A$

Idea: $\exists!$ a morphism $a \to a'$ for every pair of objects a, a'



This is what I want

▷ This is a category

(Rmk: Every morph. in $M(A)$ is an isomorphism)

$$\begin{tikzcd} 3 \arrow[rr, "h"] \arrow[dr, "g"', "\exists \text{ unique}"] & & 1 \\ & 2 \arrow[ur, "f"', "\exists \text{ unique}"] & \end{tikzcd}$$

h must be $f \cdot g$!

$$(u \cdot v) \cdot w = u \cdot (v \cdot w)$$

SAME DOMAIN & CODOMAIN