2-Categories

lecture notes for TalTech ITI9200

Spring 2021

1 Motivation

A collection of categories and the functors between them forms a category because functor composition is defined in terms of function composition, and is therefore strictly unital and associative.

Additionally, for any pair of categories $\mathbb{A}$ and $\mathbb{B}$ we have a hom category $\mathbb{A} \to \mathbb{B}$, whose objects are functors and morphisms are natural transformations. The composition of functors together with the horizontal composition of natural transformations induces a composition structure on these hom categories in the following sense.

For any three categories, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, we can define a *composition functor* $(\mathbb{A} \to \mathbb{B}) \times (\mathbb{B} \to \mathbb{C}) \to (\mathbb{A} \to \mathbb{C})$ that takes an object of $\mathbb{A} \to \mathbb{B}$ and an object of $\mathbb{B} \to \mathbb{C}$—which is to say two consecutive functors—and combines them to the object of $\mathbb{A} \to \mathbb{C}$ that is their composite functor. Likewise, we can take a pair of arrows from these categories—which is to say two adjacent natural transformations—and combine them by horizontal composition to a natural transformation between the composites of their boundary functors.

Horizontal composition of natural transformations is strictly unital and associative because it is defined in terms of the composition of morphisms in the codomain category. Thus the composition functor on hom categories is strictly unital and associative as well. The units are give by a functor from the singleton category, $\mathbb{1} \to (\mathbb{A} \to \mathbb{A})$, that picks out the identity functor and its identity natural transformation.

This gives any collection of categories, the functors between them, and the natural transformation between those a 2-dimensional categorical structure, where the categories and functors form a category, each of whose homs is itself a category in a compatible way.

2 Definition

A 2-category is a generalization of this situation, without the assumptions that the elements involved are categories, functors and natural transformations.

Definition 2.1 (2-category)
A 2-*category* $\mathbb{C}$ consists of the following *structure*:

objects: A collection of 0-cells or "objects" $\mathbb{C}_0$.

hom categories: For each pair of objects $A, B \in \mathbb{C}_0$ a hom category $\mathbb{C}(A \to B)$.

composition functors: For each triple of objects $A, B, C \in \mathbb{C}_0$, a composition functor $\kappa_{A,B,C} : \mathbb{C}(A \to B) \times \mathbb{C}(B \to C) \to \mathbb{C}(A \to C)$.

identity functors: For each object $A \in \mathbb{C}_0$ an identity functor whose domain is the singleton category $\eta_A : \mathbb{1} \to \mathbb{C}(A \to A)$.

Before stating the axioms we fix the following terminology and notation.

- We write "$A : \mathbb{C}$" for $A \in \mathbb{C}_0$.
- An object of a hom category $f : \mathbb{C}(A \to B)$ is called a 1-cell or "arrow" of $\mathbb{C}$.
- A morphism of a hom category $\alpha : \mathbb{C}(A \to B)(f \to f')$ is called a 2-cell or "globe" of $\mathbb{C}$.
- We may drop any prefix of the boundary specification of an arrow or globe when it is clear from the context. So we may write $\mathbb{C}(A \to B)$ as "$A \to B$", and $\mathbb{C}(A \to B)(f \to f')$ as "$(A \to B)(f \to f')$" or as "$f \to f'$".
- For consecutive globes in a hom category $\alpha : (A \to B)(f \to f')$ and $\gamma : (A \to B)(f' \to f'')$, we write "$\alpha \cdot \gamma$" $: (A \to B)(f \to f'')$ for their composition as morphisms of the hom category, and refer to this as "vertical composition of globes".
- For consecutive arrows $f : A \to B$ and $g : B \to C$ we write "$f \cdot g$" $: A \to C$ for their image under the functor $\kappa_{A,B,C}$, and refer to this as "composition of arrows".
- For adjacent globes $\alpha : (A \to B)(f \to f')$ and $\beta : (B \to C)(g \to g')$ we write "$\alpha \cdot\cdot \beta$" $: (A \to C)(f \cdot g \to f' \cdot g')$ for their image under the functor $\kappa_{A,B,C}$, and refer to this as "horizontal composition of globes".
- We write "$\mathrm{id}\, A$" $: A \to A$ for the image of the sole object of $\mathbb{1}$ under the functor η_A and refer to this as the "identity arrow" on A.
- We write "$\mathrm{id}^2 A$" $: (A \to A)(\mathrm{id}\, A \to \mathrm{id}\, A)$ for the image of the sole morphism of $\mathbb{1}$ under the functor η_A and refer to this as the "(double) identity globe" on A.

The structure of a 2-category must satisfy the following *properties*:

- Arrow composition is strictly unital and associative:

$$\mathrm{id}\, A \cdot f = f = f \cdot \mathrm{id}\, B \quad \text{and} \quad (f \cdot g) \cdot h = f \cdot (g \cdot h)$$

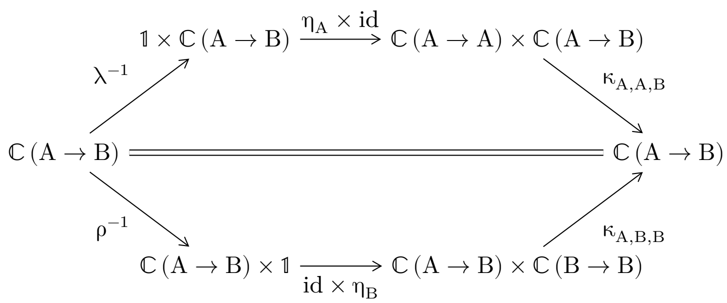
- Horizontal globe composition is strictly unital and associative:

$$\mathrm{id}^2 A \cdot\cdot \alpha = \alpha = \alpha \cdot\cdot \mathrm{id}^2 B \quad \text{and} \quad (\alpha \cdot\cdot \beta) \cdot\cdot \gamma = \alpha \cdot\cdot (\beta \cdot\cdot \gamma)$$

Note that the second property is *dependent* upon the first one in the sense that the boundaries of the globes are well-defined only if the equations between the arrows are satisfied.

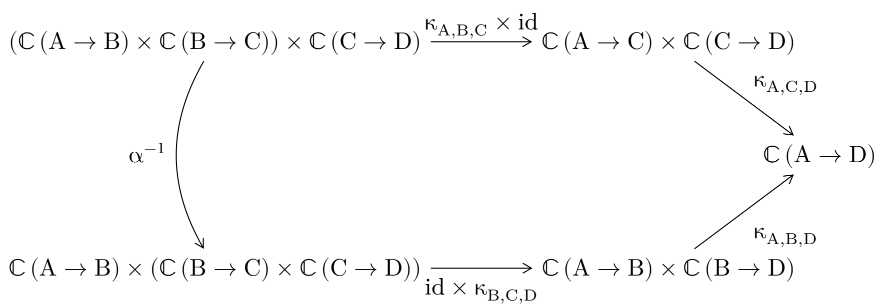
We can express these in terms of κ and η with the following commuting diagrams.

strict unitality:



where λ and ρ are the unitor natural isomorphisms for the cartesian product.

strict associativity:

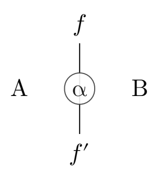


where α is the associator natural isomorphism for the cartesian product.

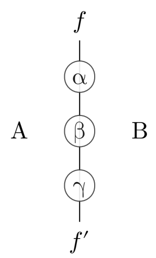
Chasing arrows and globes around these two diagrams gives us precisely the unitality and associativity we wanted.

3 Diagrams

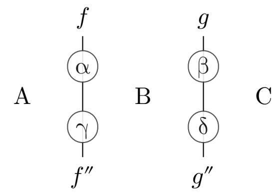
We can draw diagrams to represent globes in a 2-category. We represent a globe $\alpha : (A \to B)(f \to f')$ as follows:



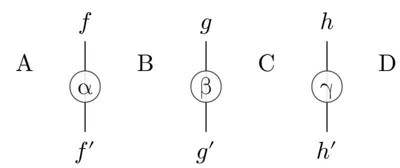
Because composition in the hom categories is strictly unital and associative, diagrams like the following are unambiguous:



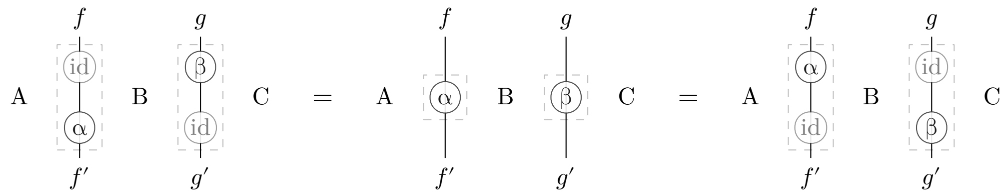
Because κ is a functor on the hom categories, diagrams like the following are unambiguous:



Because composition of arrows and horizontal composition of globes is strictly unital and associative, diagrams like the following are unambiguous:



In fact, any diagram we can draw that represents a globe of a 2-category represents a unique globe. But more is true. Because identity globes are strict units of composition in hom categories, “beads” on independent “wires” can “slide” past one another:



This lets us prove theorems about 2-categories by bead-pushing (figure 1).

4 Examples

Example 4.1 (2-categories of categories)
Any collection of categories, together with the functors between them and the natural transformation between those forms a 2-category, as described in the motivation section.

Figure 1: Category theory students collaborating on a proof.

Example 4.2 (categories as locally discrete 2-categories)
We can regard a category as a 2-category with discrete hom categories (i.e. the only globes are the identity globes).

- Arrow composition is strictly associative and unital by the category axioms.
- Horizontal composition of globes is strictly associative and unital because there is at most one globe bounded by any pair of arrows.

Example 4.3 (strict monoidal categories as 1-object 2-categories)
We can regard a strict monoidal category $\mathbb{M}$ as a 2-category $\mathbb{C}$ with a single object $\star$.

- We take the objects $\mathrm{A} : \mathbb{M}$ as the arrows $\mathrm{A} : \mathbb{C}\,(\star \to \star)$.
- We take the morphisms $\mathbb{M}\,(\mathrm{A} \to \mathrm{B})$ as the globes $\mathbb{C}\,(\star \to \star)\,(\mathrm{A} \to \mathrm{B})$.
- We take the monoidal product functor $\otimes : \mathbb{M} \times \mathbb{M} \to \mathbb{M}$ as the composition functor $\kappa_{\star,\star,\star} : \mathbb{C}\,(\star \to \star) \times \mathbb{C}\,(\star \to \star) \to \mathbb{C}\,(\star \to \star)$.
- We take the monoidal unit functor $\mathrm{I} : \mathbb{1} \to \mathbb{M}$ as the identity functor $\eta_\star : \mathbb{1} \to \mathbb{C}\,(\star \to \star)$.

The associativity and unitality laws for the 2-category $\mathbb{C}$ are justified by the respective laws of the monoidal category $\mathbb{M}$.

Example 4.4 (locally ordered 2-category of sets and relations)
The category of sets and relations extends to a 2-category by locally ordering the homs by pointwise implication.

- For relations $\mathrm{R}, \mathrm{R}' : \mathrm{A} \nrightarrow \mathrm{B}$ we say $\mathrm{R} \leq \mathrm{R}'$ if

$$\forall\, a \in \mathrm{A}, b \in \mathrm{B} \;.\; \mathrm{R}\,(a \dashrightarrow b) \supset \mathrm{R}'\,(a \dashrightarrow b)$$

- The composition of relations $\mathrm{R} : \mathrm{A} \nrightarrow \mathrm{B}$ and $\mathrm{S} : \mathrm{B} \nrightarrow \mathrm{C}$ is given by $(\mathrm{R} \cdot \mathrm{S})\,(a \dashrightarrow c) := \exists\, b \in \mathrm{B} \;.\; \mathrm{R}\,(a \dashrightarrow b) \wedge \mathrm{S}\,(b \dashrightarrow c)$.

- The identity relation on a set is equality, $\mathrm{id}\,\mathrm{A}\,(a \dashrightarrow a') := a = a'$.
- The horizontal composition of relation implications is witnessed by the fact that if $\mathrm{R} \leq \mathrm{R}' : \mathrm{A} \nrightarrow \mathrm{B}$ and $\mathrm{S} \leq \mathrm{S}' : \mathrm{B} \nrightarrow \mathrm{C}$ then,

$$\begin{array}{rl} & \mathrm{R} \cdot \mathrm{S}\,(a \dashrightarrow c) \\ := & \exists\, b \in \mathrm{B}\,.\,\mathrm{R}\,(a \dashrightarrow b) \wedge \mathrm{S}\,(b \dashrightarrow c) \\ \Rightarrow & \exists\, b \in \mathrm{B}\,.\,\mathrm{R}'\,(a \dashrightarrow b) \wedge \mathrm{S}'\,(b \dashrightarrow c) \\ =: & \mathrm{R}' \cdot \mathrm{S}'\,(a \dashrightarrow c) \end{array}$$

- The composition of relations is strictly unital and and associative because they are defined extensionally.
- the horizontal composition of relation implications is strictly unital and associative because the hom categories are thin.

More generally, any category whose hom sets can be ordered in a manner compatible with composition determines a locally ordered 2-category.

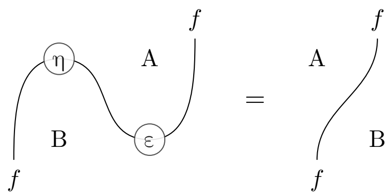
5 Constructions in 2-Categories

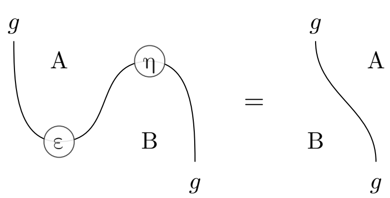
Many of the structures that we have defined in the the 2-category of categories, functors and natural transformations can be defined in an arbitrary 2-category.

Definition 5.1 (adjunction)
An *adjunction* in a 2-category $\mathbb{C}$ consists of:

- a pair of anti-parallel arrows $f : \mathbb{C}\,(\mathrm{A} \rightarrow \mathrm{B})$ and $g : \mathbb{C}\,(\mathrm{B} \rightarrow \mathrm{A})$ and
- a pair of globes $\eta : \mathbb{C}\,(\mathrm{A} \rightarrow \mathrm{A})\,(\mathrm{id} \rightarrow f \cdot g)$ and $\varepsilon : \mathbb{C}\,(\mathrm{B} \rightarrow \mathrm{B})\,(g \cdot f \rightarrow \mathrm{id})$

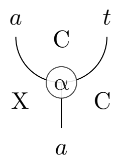
that satisfy the familiar relations:



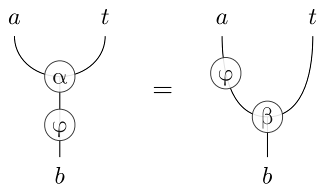


Definition 5.2 (endomorphism algebra)
For chosen objects X and C of a 2-category $\mathbb{C}$, let $t : \mathrm{C} \rightarrow \mathrm{C}$ be an endomorphism

arrow and $a : \mathrm{X} \to \mathrm{C}$ be an arrow. A *t-algebra* for a is a globe $\alpha : a \cdot t \to a$.



Definition 5.3 (endomorphism algebra morphism)
If $b : \mathrm{X} \to \mathrm{C}$ is an arrow parallel to a and $\beta : b \cdot t \to b$ is a t-algebra for b then a *t-algebra morphism* from α to β is a globe $\varphi :\ (\mathrm{X} \to \mathrm{C})\,(a \to b)$ such that:



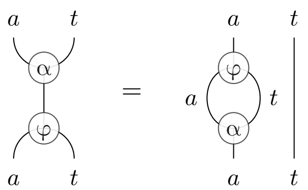
Lemma 5.4 (category of endomorphism algebras)
The collection of t-algebras and their morphisms forms a category t-$\mathrm{A{\scriptstyle LG}}_{\mathrm{X}}$ where composition of t-algebra morphisms is the vertical composition of globes in $\mathrm{X} \to \mathrm{C}$.

Lemma 5.5 (Lambek)
An initial t-algebra is an isomorphism.

Proof. Suppose that $\alpha : a \cdot t \to a$ is initial in t-$\mathrm{A{\scriptstyle LG}}_{\mathrm{X}}$.

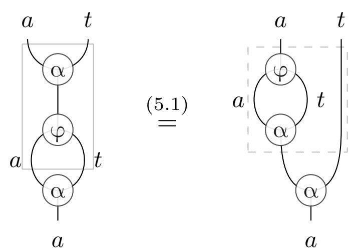
Observe that $\alpha \cdot\cdot\, t : a \cdot t \cdot t \to a \cdot t$ is a t-algebra for $a \cdot t : \mathrm{X} \to \mathrm{C}$.

By the assumption that α is initial, there is a unique globe $\varphi : a \to a \cdot t$ such that:

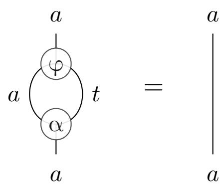


(5.1)

Next observe that the globe $\varphi \cdot \alpha : a \to a$ is a t-algebra morphism $\alpha \to \alpha$ since by post-composing equation 5.1 with α we get:

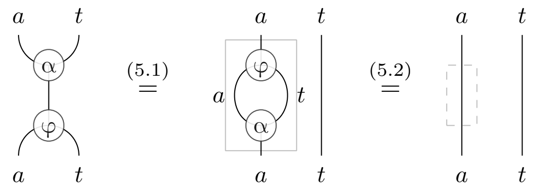


Of course, $\mathrm{id}\,\alpha$ is a t-algebra morphism, so by the initiality of α we have that φ is a section for α:



(5.2)

To see that it is a retraction as well, we again use equation 5.1 to obtain:

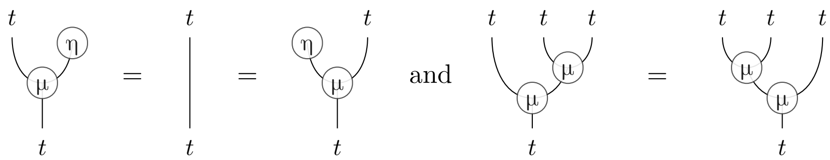


□

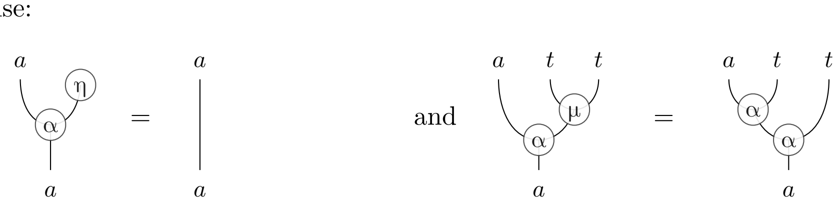
Definition 5.6 (monad)
A *monad* in a 2-category $\mathbb{C}$ consists of:

- an endomorphism arrow $t : \mathrm{C} \to \mathrm{C}$ and
- a pair of globes $\mu : t \cdot t \to t$ and $\eta : \mathrm{id}\,\mathrm{C} \to t$

that satisfy the familiar relations:

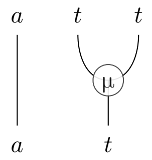


Definition 5.7 (monad algebra)
For chosen objects X and C of a 2-category $\mathbb{C}$, Let $\mathrm{M} := (t\,,\mu\,,\eta)$ be a monad on the object C and $\alpha : (\mathrm{X} \to \mathrm{C})\,(a \cdot t \to a)$ be a t-algebra for a. Then α is an M-*algebra* for a if it is compatible with the monad structure in the following sense:



Definition 5.8 (category of monad algebras)
A *monad algebra morphism* is just an endomorphism algebra morphism between monad algebras. This makes the category of algebras for a monad a full subcategory of the category of algebras for its underlying endomorphism arrow. This category is called the “Eilenberg–Moore category” of the monad.

Lemma 5.9 (free monad algebra)
For any any arrow $a : \mathrm{X} \to \mathrm{C}$ and monad $\mathrm{M} \coloneqq (t\,,\mu\,,\eta)$ on $t : \mathrm{C} \to \mathrm{C}$ there is an M-algebra for the arrow $a \cdot t$ given by:

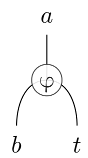


called the *free* M-*algebra* on a. Note that this terminology can be confusing because the arrow underlying the free M-algebra on a is not a but rather $a \cdot t$.

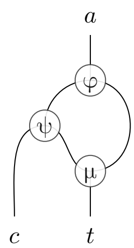
Lemma 5.10 (Kleisli category)
For any object X and any monad $\mathrm{M} \coloneqq (t\,,\mu\,,\eta)$ on $t : \mathrm{C} \to \mathrm{C}$ there is a category K given by the following data.

objects: an object $a : \mathrm{K}$ is an arrow $a : \mathrm{X} \to \mathrm{C}$.

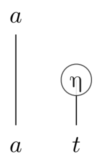
morphisms: a morphism $\varphi : \mathrm{K}\,(a \to b)$ is a globe:



composition: for $\varphi : \mathrm{K}\,(a \to b)$ and $\psi : \mathrm{K}\,(b \to c)$ the composite $\varphi \cdot \psi : \mathrm{K}\,(a \to c)$ is given by:



identities: for $a : \mathrm{K}$ the identity $\mathrm{id}\,a$ is given by:



Theorem 5.11 (challenge)
For a given object X and monad $\mathrm{M} \coloneqq (t\,,\mu\,,\eta)$ on $t : \mathrm{C} \to \mathrm{C}$, there is an isomorphism between the full subcategory of free M-algebras and the Kleisli category of M.

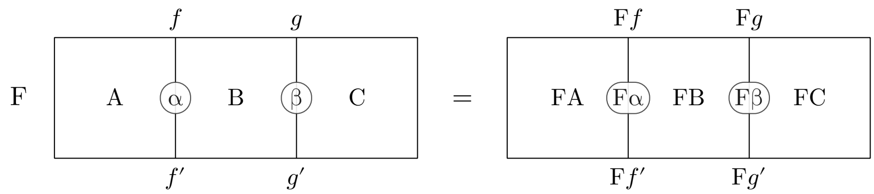
6 Morphisms of 2-Categories

As categorists, whenever we define a structure we want to know about its maps as well.

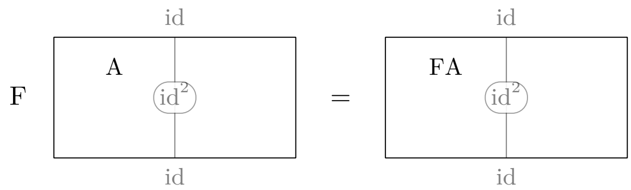
Definition 6.1 (2-functor)
A 2-*functor* between 2-categories $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ consists of the following data:

- a function on objects $\mathrm{F}_0 : \mathbb{C}_0 \to \mathbb{D}_0$,
- for each pair of objects $\mathrm{A}, \mathrm{B} : \mathbb{C}$ a local functor $\mathrm{F}_{\mathrm{A},\mathrm{B}} : \mathbb{C}\,(\mathrm{A} \to \mathrm{B}) \to \mathbb{D}\,(\mathrm{F}_0\mathrm{A} \to \mathrm{F}_0\mathrm{B})$

These functors of hom categories must strictly preserve the composition structure of arrows and the horizontal composition structure of globes:



and



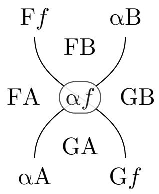
Note that F necessarily preserves the vertical composition structure of globes by virtue of the $\mathrm{F}_{\mathrm{A},\mathrm{B}}$s being functors on the hom categories.

Like a natural transformation between 1-functors between 1-categories, a transformation between 2-functors between 2-categories should have component arrows for objects. But since there is now "room" for 2-dimensional structure, we can ask that it have component globes for arrows as well. There are two possible ways to orient these component globes resulting in what are known as "lax" and "oplax" transformations,

Definition 6.2 (oplax transformation between 2-functors)
An *oplax transformation* between 2-functors between 2-categories $\alpha : (\mathbb{C} \to \mathbb{D})\,(\mathrm{F} \to \mathrm{G})$ consists of the following data.

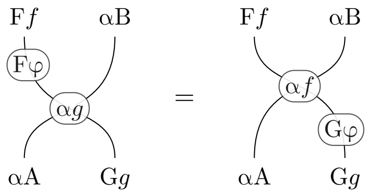
object-component arrows: for each object $\mathrm{A} : \mathbb{C}$ we have an arrow $\alpha\mathrm{A} : \mathbb{D}\,(\mathrm{FA} \to \mathrm{GA})$,

arrow-component globes: for each arrow $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ we have a $\mathbb{D}$-globe:



such that for each globe $\varphi : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})\,(f \to g)$ we have,

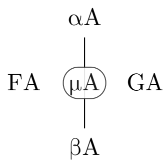
$$(\mathrm{F}\varphi \cdot\cdot\, \alpha\mathrm{B}) \cdot \alpha g \;=\; \alpha f \cdot (\alpha\mathrm{A} \cdot\cdot\, \mathrm{G}\varphi) \;:\; \mathbb{D}\,(\mathrm{FA} \to \mathrm{GB})\,(\mathrm{F}f \cdot \alpha\mathrm{B} \to \alpha\mathrm{A} \cdot \mathrm{G}g).$$



Just as the homs between categories form categories, the homs between 2-categories form 2-categories.

Definition 6.3 (modification between oplax transformations)
A *modification* between oplax transformations between 2-functors between 2-categories $\mu :\ (\mathbb{C} \to \mathbb{D})\,(\mathrm{F} \to \mathrm{G})\,(\alpha \to \beta)$ consists of the following data.

object-component globes: for each object $\mathrm{A} : \mathbb{C}$ a $\mathbb{D}$-globe:



Such that for each arrow $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ we have

$$\alpha f \cdot (\mu\mathrm{A} \cdot\cdot\, \mathrm{G}f) \;=\; (\mathrm{F}f \cdot\cdot\, \mu\mathrm{B}) \cdot \beta f \;:\; \mathbb{D}\,(\mathrm{FA} \to \mathrm{GB})\,(\mathrm{F}f \cdot \alpha\mathrm{B} \to \beta\mathrm{A} \cdot \mathrm{G}f).$$