Category theory and its applications - ITI9200

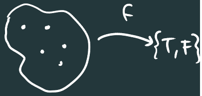
https://compose.ioc.ee

QUESTIONS ?

A worked-out example   Set / Truth = $\mathcal{C}$ Set / Truth

- Truth = $\{T, F\}$ ($\{0,1\}$, $\{\bot, \top\}$)...

▷ Objects in $\mathcal{C}$ are functions $f : A \longrightarrow \{T, F\}$



f can be regarded as a <u>predicate</u> valid on A

$A_T \subseteq A$ elements a such $f(a) = T$

$A_F \subseteq A$ $f(a') = F$

$* \Rightarrow A = A_T \sqcup A_F$

▷ Morphism

$$\begin{tikzcd} \overset{A_T \sqcup A_F}{A} \arrow[r, "h"] \arrow[dr, "f"'] & \overset{B_T \sqcup B_F}{B} \arrow[d, "g"] \\ & \{T,F\} \end{tikzcd}$$

?

$h = (h' : A_T \to B_T)$

$h'' : A_F \to B_F$ $\boxed{T}$ $\boxed{F}$ NO T F

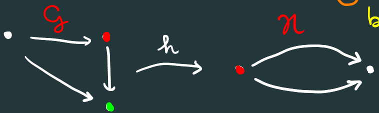
"closes the triangle"

- h is such that $\forall a \in A$ $g(h(a)) = f(a)$

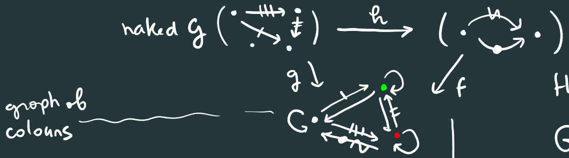
Since $*$, $a \in A_T$ $f(a) = \text{true}$

h must send a into an elem. $h(a) \in B_T$

because $g(ha) = fa = \text{true}$



Colored Graph = dGph / "colors"
dre



$A \xrightarrow{f} B$ is a monomorphism in $\mathcal{C}$

$\big($ hom$(X, _)$ is a functor $\mathcal{C} \longrightarrow$ Set $\big)$

$\forall X$, $\mathrm{hom}(X, A) \xrightarrow{f_*} \mathrm{hom}(X, B)$ is injective

$(u : X \to A) \longmapsto (f \circ u : X \to B)$

If $\left\{ \begin{array}{l} \mathrm{hom}(X, A) \\ (\mathrm{hom}(X, B) \end{array} \right.$ is not a set?

Homework

Giving h as a morphism in the slice cat $\mathrm{dGph}/\text{graph of colours}$

$\equiv$ Giving $h : \mathcal{G} \to \mathcal{H}$ (naked graphs)

+ choice of colorings on $\mathcal{G}, \mathcal{H}$

+ fact h preserved said coloring

Proposition $p : E \longrightarrow B$ morphism in $\mathcal{C}$ a category

p is epimorphism [1] ($u \cdot p = v \cdot p \Rightarrow u = v$)

& p is split monomorphism [2] ($s : B \longrightarrow E$ such that $s \cdot p = 1$)

(p has a left inverse) $\overset{2}{}s \cdot p = 1_E$

$\Rightarrow$ p is monomorphism

$\Rightarrow$ p is an isomorphism

(since inverses are unique $p^{-1} = s$)

Proof: GOAL $p \cdot s = 1_B$.

$p \cdot s \cdot p = p \cdot 1_E$

What do we know? 1) & 2). Using 2) $s \cdot p = 1_E$ $\leadsto$ $p \cdot s \cdot p = p$.

Since 1) $(p \cdot s) \cdot p = 1_B \cdot p$ $\quad p \cdot s = 1_B$ QED!

Def: $r : A \longrightarrow B$ in $\mathcal{C}$, split epimorphism $\exists\ s : B \to A$ such that $r \cdot s = 1_B$
($s =$ "right inverse")

If we are given the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, "u"'] \arrow[dr, phantom, "\text{comm.}"] & A \arrow[r, "r"] \arrow[d, "v"] \arrow[dr, phantom, "\text{comm.}"] & B \arrow[d, "u"] \\
B' \arrow[r, "s'"'] \arrow[urr, dashed] & A' \arrow[r, "r'"'] \arrow[u, bend left, "v^{-1}"] & B'
\end{tikzcd}$$

r is split epi $\to (r \cdot s = 1_B)$
r' is split epi $(r' \cdot s' = 1_{B'})$

- If v isomorphism, then so is u.

Proof We know
- v invertible
- r, r' are split epi (retracts)

$\Rightarrow$ $B \xrightarrow{u} B'$ is invertible

? How to get an arrow $B' \longrightarrow B$?

$g = \underline{r \cdot v^{-1} \cdot s'}$ is a candidate

? $g \cdot u = 1$, $u \cdot g = 1$

$g \cdot u = r \cdot v^{-1} \cdot \underline{s' \cdot u}$
$= r \cdot \underline{v^{-1} \cdot v} \cdot s$
$= r \cdot s = 1$

$u \cdot g =$ idea is the same. FILL IN!

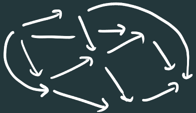
The free category over a digraph

$$\mathcal{G}\begin{cases} G_0 & \text{vertices} \\ G_1 & \text{edges} \end{cases} \qquad s,t : G_1 \rightrightarrows G_0$$

$$\begin{array}{ccc} \cdot & \overset{s}{\longmapsto} & \cdot \\ \downarrow & & \\ \cdot & \underset{t}{\longmapsto} & \cdot \end{array}$$

a category, built out of the graph $\mathcal{G}$ with the property that

"no nontrivial commutativity equations hold in it"



no nontrivial relations between "abstract composition of morphisms".

—— ∘ ——

$\mathcal{G} \longmapsto F\mathcal{G}$

$(G_0, G_1) \longmapsto$ objects of $F\mathcal{G}$ can only be G_0

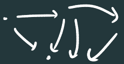
morphisms "are" the edges of $\mathcal{G}$:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "{(g,f)}"'] & y \arrow[r, "g"] & z \end{tikzcd}$$

this should have a composition!

(Note: this is similar to another construction, the free monoid (list monoid) on a set)

Def: the free category on a directed graph G has

objects = the vertices of G 

morphisms $x \in G_0 \longrightarrow y \in G_0$

$$\hom_{FG}(x,y) := \left\{ x \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} a_n \xrightarrow{f_{n+1}} y \;\middle|\; \begin{array}{l} n \geq 0 \\ a_1 - a_n \in G_0 \\ f_1 - f_{n+1} \in G_1 \end{array} \right\}$$

"can be composed"

1 — composition is just concatenation of chains

2 — identities are empty chains in each $\hom_{FG}(x,x)$ there exists an element "( )" which plays the role of the ident. of x

empty chain

- Composition is associative

$$(w \cdot v) \cdot u$$

$$(x \xrightarrow{u} y,\; y \xrightarrow{v} z,\; z \xrightarrow{w} t)$$

- Identit. are really identities

$$w : z \to c_1 \to \cdots \to c_r \to t$$

$$v : y \to b_1 \to \cdots \to b_q \to z$$

$$u : x \to a_1 \to \cdots \to a_p \to y$$

$(w \cdot v) \cdot u$ is (

$$\underbrace{x \to a_1 \to \cdots \to a_p \to y}_{u}\ \underbrace{\to b_1 \cdots b_q \to z}_{v}\ \underbrace{\to c_1 \to \cdots \to c_n \to t}_{w}$$

$w \cdot (v \cdot u)$ is

$$(x \to \cdots \to \cdots \to y \to \cdots \to \cdots \to z \to \cdots \cdots \to t)$$

__Examples__:

$G_1 = \{ \bullet \, , \text{ no edges} \}$

FG_1 has a single object, $\bullet$
only the empty sequence!

$FG_1 = \{ \bullet \circlearrowleft \mathrm{id} \}$ (there are no edges to build longer chains!)

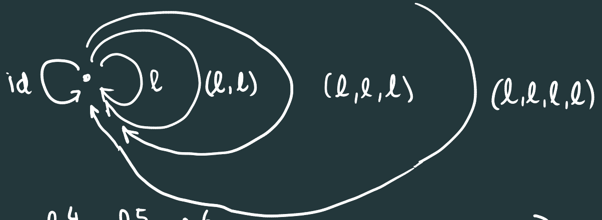
$G_2 = \{ \bullet \circlearrowleft l \}$

FG_2 has objects just $\bullet$

$\mathrm{id} \circlearrowleft \bullet \circlearrowleft (l), \quad l, \quad (l,l)$

$l \cdot l = (l,l)$ new element of FG_2

FG_2



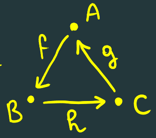
$$\{ id, \ell, \underset{(\ell,\ell)}{\ell\cdot\ell}, (\ell,\ell,\ell), \ell^4, \ell^5, \ell^6, \dots \}$$

$$\{ 0, 1, 2, 3, 4, 5, 6, \dots \} = \mathbb{N} \,!$$

$G_2 = \{ \bullet \circlearrowleft \}$

FG_2 is "natural numbers"! ▨

$G_3 =$



$FG_3 =$