Category theory and its applications - ITI9200

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New categories from old ones

SUBSTRUCTURES $\cdot\ (\mathbb{N}, \leq) \supseteq (2\mathbb{N}, \leq)$

$\cdot$ SUBMONOIDS $(M, \cdot)$ subset $N \subseteq M$ $n, n' \in N$

$n \cdot n' \in N$

Definition (Subcategory)
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- for every morphism $-\boxed{f}-$ in hom(S), both the source X and the target Y are in $\mathrm{ob}(\mathcal{S})$,
- for every pair of morphisms $\overset{X}{-}\boxed{f}\overset{Y}{-}$ and $\overset{Y}{-}\boxed{g}\overset{Z}{-}$ in $\mathrm{hom}(\mathcal{S})$ the composite $\overset{X}{-}\boxed{f\,;g}\overset{Z}{-}$ is in $\mathrm{hom}(\mathcal{S})$ whenever it is defined.

$\mathcal{C}$ = Sets & functions

finite sets, all functions

all sets, just <u>injective</u> functions.

CARTESIAN PRODUCT of sets : A, B $A \times B$

$a, a' \ldots$ (a, b) $(a', b) \ldots$

$b, b' \ldots$ (a', b') $(a, b') \ldots$

Definition (Product of two categories)
Let $\mathcal{C}, \mathcal{D}$ be categories; the product category $\mathcal{C} \times \mathcal{D}$ has

$\mathcal{C}_0$ objects of $\mathcal{C}$
$\mathcal{D}_0$ " " $\mathcal{D}$

- objects the pairs (C, D) of objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$;
- morphisms $(C, D) \to (C', D')$ the pairs (f, g) of morphisms $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$.

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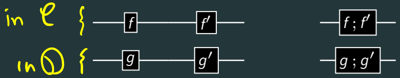
Composition and identity act in parallel, in $\mathcal{C}$ and $\mathcal{D}$ separately.



(Only happens at the level of morphism)

(dual)

opposite of $\mathcal{C}$

Definition (Opposite category)
Let $\mathcal{C}$ be a category; then, we can define a new category $\mathcal{C}^{\text{op}}$ out of $\mathcal{C}$, having the same objects as $\mathcal{C}$, and where there is an arrow

$A \xrightarrow{\boxed{f}} B$ if and only if there is an arrow $B \xrightarrow{\boxed{f}^{op}} A$.

$$\mathcal{C} = \left\{ \begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd} \right\}$$

$$\mathcal{C}^{op} = \left\{ \begin{tikzcd} A & B \arrow[l, shift right, "f^{op}"'] \arrow[l, shift left, "g^{op}"] & C \arrow[l, "h^{op}"'] \end{tikzcd} \right\}$$

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$\left(A \xrightarrow{f} B \xrightarrow{g} C\right)$ in $\mathcal{C}$ $\dashrightarrow$ $C \xrightarrow{g^{op}} B \xrightarrow{f^{op}} A$ in $\mathcal{C}^{op}$

can be composed in $\mathcal{C}$: $A \xrightarrow{f;g} C$ $\dashrightarrow$ unique $C \xrightarrow{g^{op};f^{op}} A$ is the composition of

$\mathbb{N}^{op}$

- the poset $\mathbb{N}$ ordered by $\geq := (\leq)^{\text{op}}$ is just the opposite order $\{\cdots \leq 3 \leq 2 \leq 1 \leq \underline{0}\}$; hom$(n, m)$ is $\neq \emptyset$ if $n \leq m$
- the poset $\mathbb{Z}$ ordered by $\geq := (\leq)^{\text{op}}$ is just $\mathbb{Z}$ itself (right?);
- the monoid $(M^{\text{op}}, \cdot^{\text{op}})$ is just the opposite monoid with multiplication $x \cdot^{\text{op}} y := y \cdot x$.

$n \circlearrowleft \bullet \circlearrowright m$

monoids = cats w/ single object

↳ opposite of M is the opposite cat of the monoid regarded as a category

Definition (Slice categories)
Let $\mathcal{C}$ be a category; fix an object $B \in \mathcal{C}_0$; then, there is a category $\mathcal{C}/B$ (‘$\mathcal{C}$ over B’) whose

- objects are all morphisms in $\mathcal{C}$ with codomain B;
- morphisms are $h : X \to Y$ that are ‘over B’, in the sense that

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[r, dotted, "k"] & {} \\ & B & & \end{tikzcd}$$

$g \cdot h = f$

$r\, k\, h \overset{2}{=} g h \overset{1}{=} f$

- assoc.
- identies (identies $\begin{tikzcd} X \arrow[rr, equal, "id"] \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$)

$$\hom_{\mathcal{C}/B}(f, g) = \left\{ s(f) \;\text{—}\boxed{h}\text{—}\; s(g) \;\middle|\; \text{—}\boxed{h\,;g}\text{—} \;=\; \text{—}\boxed{f}\text{—} \right\}$$

Definition (Arrow and cube category)

- the arrow category $\mathcal{C}^{\to}$ has objects the arrows of $\mathcal{C}$, and morphisms $(u, v) : f \to g$ the commutative squares

(u, v)

$u : A \longrightarrow C$

$v : B \longrightarrow D$

$+ \quad f ; v = u ; g$

$$\begin{tikzcd} A \arrow[r, dotted, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\rightsquigarrow"] & C \arrow[d, "g"] \\ B \arrow[r, dotted, "v"'] & D \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \bullet \arrow[r, "u'"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "v"'] & \bullet \arrow[r, "v'"'] & \bullet \end{tikzcd}$$

$\Downarrow$

$$\begin{tikzcd} \bullet \arrow[r, "u ; u'"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "v ; v'"'] & \bullet \end{tikzcd}$$

$h u' u = v' g u$

$= v' v f$. $\blacksquare$

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$$\begin{tikzcd} {} \arrow[r, dotted, "u"] \arrow[d, "f"'] & {} \arrow[d, "g"] \\ {} \arrow[r, dotted, "v"'] & {} \end{tikzcd}$$

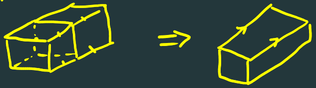
cat of squares

- the cube category $\mathcal{C}^{\square}$ has objects the squares as above, and morphisms the commutative cubes

$$\begin{tikzcd} & A' \arrow[rr] \arrow[dd] & & C' \arrow[dd] \\ A \arrow[rr] \arrow[dd] & & C \arrow[dd] & \\ & B' \arrow[rr] & & D' \\ B \arrow[rr] & & D & \end{tikzcd}$$

$$\begin{tikzcd} & {} \arrow[rr] \arrow[dd] & & {} \arrow[dd] \\ {} \arrow[ur, "u_1"] \arrow[rr] \arrow[dd] & & {} \arrow[ur, "u_2"'] \arrow[dd] & \\ & {} \arrow[rr] & & {} \\ {} \arrow[ur, "u_3"] \arrow[rr] & & {} \arrow[ur, "u_4"'] & \end{tikzcd}$$

+ every face of the cube is a comm. square



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$$\begin{tikzcd} & {} \arrow[rr] \arrow[dd] & & {} \arrow[dd] \\ {} \arrow[ur, dotted] \arrow[rr] \arrow[dd] & & {} \arrow[ur, dotted] \arrow[dd] & \\ & {} \arrow[rr] & & {} \\ {} \arrow[ur, dotted] \arrow[rr] & & {} \arrow[ur, dotted] & \end{tikzcd}$$

SLICE some morphism

Can you fill in the details? Can you regard $\mathcal{C}/B$ as subcategory of $\mathcal{C}^{\to}$, if $g = 1_B$?

Classes of morphisms in a category

E.G. in the cat of sets & functions, $f : A \longrightarrow B$

- can be bijective (invertible) $\exists\, g : B \to A$ $\begin{cases} f \cdot g = id_B \\ g \cdot f = id_A \end{cases}$
- can be injective $f(a) = f(a') \implies a = a'$
- can be surjective every $b \in B$ is fa for some $a \in A$

Recall that an isomorphism in $\mathcal{C}$ is a morphism A —$\boxed{f}$— B such that there exists a B —$\boxed{g}$— A with the property that $f\,;\underline{g} = 1_A$ and $\underline{g}\,; f = 1_B$.

Given f, if g exists g is unique: assume there are two g, g' with the same property as above, relative to f. Then

$$g' = 1g' = (gf)g' = gfg' = g(fg') = g1 = g$$

if $f : A \longrightarrow B$ has a LEFT inverse g
& RIGHT inverse g' $\Rightarrow$ f is invertible
$g = g'$

The unique g such that $gf = 1$, $fg = 1$ is called the inverse of f, and denoted f^{-1}.

Incidentally, this showed a bit more.

Isomorphisms satisfy the 2-out-of-3 property: given a commutative triangle

$$
\begin{tikzcd}
A \arrow[r, "gf"] \arrow[d, "f"'] & B \arrow[l, bend right, "u"'] \arrow[dl, bend left, "w"] \\
C \arrow[ur, "g"'] \arrow[u, bend left, "v"]
\end{tikzcd}
$$

gf, f iso $\Rightarrow g$ iso

fu is candidate g^{-1}

$gfu = 1$ u, v are inverses of gf

$fug = 1$ $\rightarrow$ (. . .)

FILL IN.

if 2 among f, g, gf are isomorphisms then so is the third.

- composition of isomorphisms is an isomorphism $\boxed{\checkmark}$ f, g iso $(gf)^{-1} = f^{-1} \cdot g^{-1}$
- Isomorphisms can be canceled.

$m: A \longrightarrow B \qquad m(x) = m(y) \Rightarrow x = y$ (elts of A)

Let $\mathcal{C}$ be the category of sets; then, the following are equivalent for $m : A \to B$:

1 • m is an injective function;

2 • for every $f, g : X \to A$, one has $mf = mg \Rightarrow f = g$. $X \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} B$

$mf = mg \longrightarrow f = g$

$2 \Rightarrow 1 \quad X = \{\bullet\} \qquad \{\{\bullet\} \xrightarrow{x} A\} = A$ (elements of A)

$1 \Rightarrow 2$ because if $mf = mg$, $\forall x \in X \quad m(\underbrace{f(x)}_{\in A}) = m(\underbrace{g(x)}_{\in A}) \overset{1}{\Longrightarrow} fx = gx$

functions coinciding on every point are the same function!

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In a category,

Definition (Monomorphism)
Let $\mathcal{C}$ be any category; 

is a monomorphism if $m : A \to B$

$$\underline{mf = mg} \Rightarrow \underline{f = g}$$

"monomorphism" = monic arrow
morphism

Proposition

1 • Identity morphisms are monic. ✓ $f = 1 \cdot f = 1 \cdot g = g$
(isomorphisms are all monomorphisms)

2 • Composites of monics are monic.

3 • If the composite $m \,;\, n$ is monic then so is m.

Proof of 2

$A \xrightarrow{m} B$, $B \xrightarrow{n} C$ m, n are monomorphism

$X \overset{f}{\underset{g}{\rightrightarrows}} A \xrightarrow{n \cdot m} C$

then $n \cdot m : A \to C$ is a monom.

if $n \cdot m \cdot f = n \cdot m \cdot g \to f = g$

n mono $\to mf = mg \to$ m mono

of $\mathcal{C}$ over A

If we take the subcategory of a slice category containing just the monomorphisms then we get a preorder category: any $f : M \to N$ must be monic itself, and must be unique because n is monic.

$(\mathrm{Set}/A)_{\text{monos}}$

$$
\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, hook, "m"'] & {} \arrow[d, phantom, "="] & N \arrow[dl, hook', "n"] \\
& A &
\end{tikzcd}
$$

$$n \cdot f = n \cdot g \xrightarrow{n \text{ mono}} f = g$$
$$\overset{\shortparallel}{m} \qquad \overset{\shortparallel}{m}$$

Monics $m : M \hookrightarrow A$ behave like subsets of a set, and represent propositions valid for A. The presence of an f as above is a way to express categorically logical entailment.

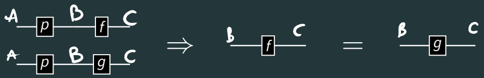
Using a preorder to interpret the entailment relation on propositions sacrifices proof relevance: it lets us say that one proposition entails another, but not why it does so.

$f : A \to B$ surjective ; $\forall b \in B \ \exists a \in A : f(a) = b$

Epimorphism are the dual concept; formally, they really behave the same way. an epimorphism in $\mathcal{C}$ $\equiv$ a monomorphism in $\mathcal{C}^{op}$

OTOH particular instances of epimorphisms are sometimes an elusive concept.

Definition (Epimorphisms)
Let $\mathcal{C}$ be any category; $A \xrightarrow{p} B$ is an epimorphism if



$$fp = gp \Rightarrow f = g$$

Intermezzo: duality

Metatheorem 1.10.2 *(Duality principle) Suppose the validity, in every category, of a statement expressing the existence of some objects or morphisms or the equality of some composites. Then the "dual statement" is also valid in every category; this dual statement is obtained by reversing the direction of every arrow and replacing every composite* $f \circ g$ *by the composite* $g \circ f$.

Proof If $\mathcal{S}$ denotes the given statement and $\mathcal{S}^*$ denotes its dual statement, proving the statement $\mathcal{S}^*$ in a category $\mathscr{A}$ is equivalent to proving the statement $\mathcal{S}$ in the category $\mathscr{A}^*$, and this is supposed to be valid. □

For example, the notion of $f\colon A \longrightarrow B$ being a monomorphism in $\mathscr{A}$ means

$$\forall C \in \mathscr{A} \;\; \forall g, h \in \mathscr{A}(C, A) \;\; f \circ g = f \circ h \Rightarrow g = h.$$

The dual notion is thus that of a morphism $f\colon B \longrightarrow A$ which satisfies

$$\forall C \in \mathscr{A} \;\; \forall g, h \in \mathscr{A}(A, C) \;\; g \circ f = h \circ f \Rightarrow g = h$$

... which is exactly the notion of an epimorphism.

Examples of monics and epics

$mf = mg$

$u\cancel{m}f = u\cancel{m}g \Rightarrow f = g$

$m : A \longrightarrow B$ with a leftinverse $u : B \longrightarrow A$: $um = 1_A \Rightarrow m$ is monic.

- a function is a monomorphism if and only if it is injective; what is a split monomorphism between sets? Apparently all monics split; but…
- an homomorphism of monoids (groups, abelian groups…) is a monomorphism if and only if it is injective;
- a morphism $f : A \to B$ is a monomorphism in $\mathcal{C}$ if and only if 'postcomposing with f' is an injective function for every other object $X \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[d, "u"'] \arrow[dr, "fu"] & \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$$\begin{array}{ccc} \hom(X, A) & \longrightarrow & \hom(X, B) \\ \left[\begin{array}{c} X \\ u\downarrow \\ A \end{array}\right] & \longmapsto & \left[\begin{array}{c} X \\ fu\downarrow \\ B \end{array}\right] \end{array}$$

$u \mapsto f.u$ injective

$f.u = f.u' \Rightarrow u = u'$

! just def of f being monic

Examples of monics and epics

- a function is an epimorphism of sets if…way more complicated; a whole hierarchy of epis exists.
- an homomorphism of monoids (groups, abelian groups…) is an epimorphism if and only if it is a surjective homomorphism. Yet, fewer epimorphisms are split.

$p : A \to B$ has a right inverse: $ps = 1 \longrightarrow p$ must be epimorphism

- The inclusion i of $\mathbb{Q}$ in $\mathbb{R}$ is an epimorphism of topological spaces;

$$\mathbb{Q} \xhookrightarrow{i} \mathbb{R} \underset{g}{\overset{f}{\rightrightarrows}} X \text{ top } \Big\} \quad f|_{\mathbb{Q}} = g|_{\mathbb{Q}} \Rightarrow f = g$$

- a morphism $f : A \to B$ is an epimorphism in $\mathcal{C}$ if and only if 'precomposing with f' is an injective function:

$uf = vf \Rightarrow u = v$

$$\hom(B, X) \longrightarrow \hom(A, X)$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, bend right] & B \arrow[d, "u"] \\ & X \end{tikzcd} \qquad \begin{bmatrix} B \\ u\downarrow \\ X \end{bmatrix} \longmapsto \begin{bmatrix} A \\ uf\downarrow \\ X \end{bmatrix}$$

$- \circ f$ must be injective

A little lemma

Proposition
An isomorphism is both monic and epic. A split epimorphism that is monic must be an isomorphism.

Proof.
The first statement is obvious.

Let $p : E \to B$ be a split epimorphism: this means there is a $u : B \to E$ such that $pu = 1_B$. But now $pup = p$, and using that p is also monic, one gets $up = 1_E$. $\square$

Draw a graphical proof. Dualise the statement (it's very easy).

Bonus: a deeper example

This example contains the germs of quite a lot categorical thinking you will encounter later.

The axiom of choice asserts that given a family of nonempty sets $(A_i \mid i \in I)$ one can pick an element a_i from each x_i.

Such a family of sets can be represented as a function over the indexing set I: $p : A \to I$ (where A is the union of all A_i);

p is a split epimorphism iff there is a function $\alpha : I \to A$ picking an element a_i from each A_i in such a way that $p(\alpha(i)) = p(a_i) = i$.

The axiom of choice amounts to the request that such an α always exists;

Since every epimorphism $p : E \to B$ can be presented as a 'bundle map' above,

Fact
The axiom of choice amounts to the request that every epimorphism in Set is a split epimorphism.

Can be false elsewhere: monoids, rings.