Category theory and its applications - ITI9200

https://compose.ioc.ee

Questions?

1. FUNCTORS
2. COALGEBRAS
3. [UNIV. COALGEBRA FOR A CERTAIN FUNCTOR (Stream)]
   ($\exists$ & is unique)

A^∞ is the terminal object of the cat of Σ-coalgs

Def $\mathcal{C}$ category, $T \in \mathcal{C}_0$ is called terminal if every other A in $\mathcal{C}_0$ has a unique arrow to T
$f : A \longrightarrow T$

Functor on the category of sets $\Sigma : S \longmapsto \mathrm{Maybe}(A \times S)$

$\{\bot\} \cup (A \times S)$

$s \longmapsto (a, s')$ (A context

$S \xrightarrow{\sigma} \Sigma S$

$s \longmapsto (a, s')$ ✓ SUCCESS
$\bot$ ✗ FAIL

s' is a successive state in which the system can be (after the computation described by σ succeeded).

$\Longrightarrow$ I can apply σ to s'! If $\sigma(s')$ does not fail, $\sigma(s') \in A \times S$
(a_1, s_2)

I can apply σ to s_2 $\left\langle \begin{array}{l} \bot \\ (a_3, s_3) \end{array} \right.$

$\sigma : S \longrightarrow \mathrm{Maybe}(A \times S)$

$A^\infty = \underset{\text{finite}}{A^*} \cup \underset{\text{infinite}}{A^{\mathbb{N}}}$

"compit" $n : A^\infty \longrightarrow \text{Maybe}(A \times A^\infty)$

$() \longmapsto \perp$

$(a : as) \longmapsto (a, as)$

$\mathbb{N} = \{0\ 1\ 2\ 3 \ldots\}$

$\text{sub} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ $\rightsquigarrow$ $\text{sub} : \mathbb{N} \times \mathbb{N} \longrightarrow \text{Maybe}\ \mathbb{N}$

$(3, 1) \longmapsto 3 - 1 = 2$

$(3, 7) \longmapsto 3-7$ not an elem of $\mathbb{N}$.

$\text{sub}\ n\ m = \begin{cases} n - m & \text{if } m \le n \\ \perp \text{ "Nothing"} & \text{ow} \end{cases}$

$A \longmapsto \text{Maybe}\ A$ is a functor

$A \longmapsto A \cup \{\perp\}$

$$\begin{array}{ccc} \text{Set} & \longrightarrow & \text{Set} \\ A & \longmapsto & A + 1 \\ f \downarrow & & \downarrow \\ B & & B + 1 \end{array}$$

$a \mapsto fa$

Exercises on Functors

- $\mathcal{C}, \mathcal{D}$ cats $\quad \mathcal{C}\times\mathcal{D} \rightsquigarrow$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] & \mathcal{C}\times\mathcal{C} \\ C \arrow[d, "f"'] \arrow[r, mapsto] & (C,C) \arrow[d, "f \ \downarrow \ f"] \\ D \arrow[r, mapsto] & (D,D) \end{tikzcd}$$

GENERIC ARROW

$$\mathcal{C} = \{ 0 \xrightarrow{u} 1 \} \qquad \mathcal{C}_0 = \{0,1\} \qquad \mathcal{C}_{mor} = \{ id_0, id_1, u \}$$

$\mathcal{C}\times\mathcal{C}$

$$(\mathcal{C}\times\mathcal{C})_0 = \{(0,0),(0,1),(1,1),(1,0)\}$$

$$\begin{tikzcd} 0 \arrow[d, "id"'] & 0 \arrow[d, "u"] \\ 0 & 1 \end{tikzcd}$$

$$\begin{tikzcd} (0,0) \arrow[r, "{(id,u)}"] \arrow[d, "{(u,id_0)}"'] \arrow[dr, dashed, "*"] & (0,1) \arrow[d, "{(u,id_1)}"] \\ (1,0) \arrow[r, "{(id_1,u)}"'] & (1,1) \end{tikzcd}$$

Exercise 4 u
Show the square $*$ is commutative
(& there are identities!)

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\Delta"] & & \mathcal{C}\times\mathcal{C} & \\ 0 \arrow[d, "u"'] \arrow[r, mapsto] & (00)=(00) \arrow[r] \arrow[d] & (01) \arrow[d] \\ 1 \arrow[rr, mapsto, bend right] & (10) \arrow[r] & (11) \end{tikzcd}$$

Where does u go?

... It's (u,u)!

Δ is functor

1) $\Delta(id_A) = id_{\Delta(A)}$ ← identities go to identies

2) $\Delta(f\circ g) = \Delta(f)\circ\Delta(g)$ ← compos go to compos.

1) $\Delta(id_A) = \underbrace{(id_A, id_A)}_{\mathcal{C}\times\mathcal{C}} : \underset{\Delta A}{\overset{\|}{(A,A)}} \longrightarrow \underset{\Delta A}{\overset{\|}{(A,A)}}$

$\left(\mathcal{C} = \{0 \xrightarrow{u} 1\},\ A = 0,1\right)$

$(0,0) \xrightarrow{(u,u)} \underline{\underline{(1,1)}} \xrightarrow[\underline{\underline{(id,id)}}]{} \underline{(1,1)}$

$\mathcal{C} \xrightarrow{\Delta} \underline{\mathcal{C}\times\mathcal{C}}$;

$$\begin{tikzcd} \Delta(id_A) : (A,A) \arrow[r, "(id,id)"] & (A,A) \\ (X,Y) \arrow[u, "(f,g)"] \arrow[ur, "(f,g)"'] & \end{tikzcd}$$

$\Delta(f \cdot g)$

$\left.\begin{array}{l} f : A \longrightarrow B \\ g : C \longrightarrow A \end{array}\right\}$

$$\left[\begin{tikzcd} C \arrow[r, "g"] \arrow[dr, bend right] & A \arrow[d, "f"] \\ & B \end{tikzcd}\right] \overset{\Delta}{\longmapsto} \begin{tikzcd} (C,C) \arrow[r, "(g,g)"] \arrow[dr, bend right, "(f\cdot g, f\cdot g)"'] & (A,A) \arrow[d, "(f,f)\ \Delta(f)"] \arrow[l, phantom, "\Delta(g)" above right] \\ & (B,B) \end{tikzcd}$$

$\Delta(f \cdot g) = \Delta f \cdot \Delta g$ $\blacksquare$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] & & \mathcal{D} \\ C \arrow[rr, mapsto] \arrow[d, "u"'] & & FC \arrow[d, "Fu"'] \\ C' & & FC' \end{tikzcd}$$

1) identies $F(1_C) = 1_{FC}$

2) composition $F(\underset{\text{in } \mathcal{C}}{u \cdot v}) = Fu \underset{\text{in } \mathcal{D}}{\cdot} Fv$

$\triangleright$ if u is an isomorphism, then Fu is an isomorphism.

Proof $u : A \longrightarrow B$ is iso $=$ $u^{-1} : B \longrightarrow A$

$$\begin{cases} 1)\ u \cdot u^{-1} = 1_B \\ 2)\ u^{-1} \cdot u = 1_A \end{cases}$$

$Fu : FA \longrightarrow FB$. Let's find a morphism $FB \longrightarrow FA$, and show it is the inverse of Fu.

$F(u^{-1}) : FB \longrightarrow FA$

$F(1)$: $F(u \cdot u^{-1}) = F(1_B)$

1') $Fu \cdot F(u^{-1}) = 1_{FB}$

$F(2)$: $F(u^{-1} \cdot u) = F(1_A)$

2') $F(u^{-1}) \cdot Fu = 1_{FA}$

1'), 2') $\Rightarrow (Fu)^{-1} = F(u^{-1})$

When inverses exist are unique

$F : \mathcal{C} \longrightarrow \mathcal{D}$ preserves isomorphisms. (u iso $\Longrightarrow$ Fu iso)

$\rightarrow$?) Is it true that if Fu is isomorphism, u is an isom?

$$\underset{\text{generic arrow}}{\mathcal{C}} = \left[\begin{tikzcd} 0 \arrow[d, "u"] \\ 1 \end{tikzcd}\right] \xrightarrow{\quad F \quad} \left\{\begin{tikzcd} * \arrow[loop right, "id_*"] \end{tikzcd}\right\}$$

$0 \longmapsto *$

$1 \longmapsto *$

$\left\{\begin{array}{l} id_0 \longmapsto id_* \\ id_1 \longmapsto id_* \end{array}\right.$

$\boxed{u \longmapsto id_*}$

$Fu = id_*$ identities are iso!

(but u is not an isom.
e.g. because there is no arrow $1 \longrightarrow 0$)

Def $F : \mathcal{C} \longrightarrow \mathcal{D}$ REFLECTS isomorphisms if

Fu is an isomorphism $\Longrightarrow$ u is an isomorphism.

Example $\mathcal{C}/A \xrightarrow{\quad U \quad} \mathcal{C}$

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

sends objects to their domains
morphism to themselves

U reflects isos

$U(h)$ is an isom.
$\|$
h is an isom. ∎

1') $\underbrace{Fu}_{f} \cdot \underbrace{F(u^{-1})}_{g} = 1$

2') $\underbrace{F(u^{-1})}_{g} \cdot \underbrace{Fu}_{f} = 1$

Recall that when inverses exist they are unique!

$\Downarrow$

1') $f \cdot g = 1$

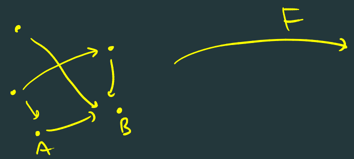
2') $g \cdot f = 1$

$$f^{-1} = g$$

$$\boxed{(Fu)^{-1} = F(u^{-1})}$$

Recall $r : A \longrightarrow B$ is a "retract" if there exists $s : B \longrightarrow A$ such that $r \cdot s = 1$ (r has a right inverse)

↳ Very same proof (half of the proof above) shows that if r is a retract, $F : \mathcal{C} \to \mathcal{D}$, Fr is a retract. [Convince yourself.].



Next Monday (Feb 15): 1st sheet of exercises

- Due after 1 week. (22 Feb)
- Email solutions 4 marking