Category theory and its applications - ITI9200

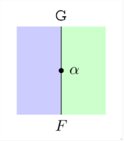
Natural Transformations   Exercises

February 17, 2021

Natural Transformations: Formal definition

Definition

Given two functors $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ (jargon: *parallel* functors), a natural transformation $\alpha : F \Rightarrow G$ is defined by:



$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathscr{D} \end{tikzcd}$$

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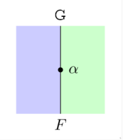

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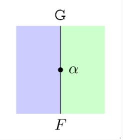

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- A morphism $\alpha_c : Fc \to Gc$ in $\mathscr{D}$, for every object $c : \mathscr{C}$, called the *components* of α
- such that for every morphism $f : c \to c'$ in $\mathscr{C}$, the following square commutes (*naturality condition*):

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \end{tikzcd}$$

Warm up: a natural transformation

Consider the finite powerset functor $\mathcal{P}_{\mathsf{fin}} : \mathsf{Set} \to \mathsf{Set}$

$$\begin{aligned} \mathcal{P}_{\mathsf{fin}}(X) &= \{U \mid U \subseteq X \text{ and } U \text{ is finite}\} \\ \mathcal{P}_{\mathsf{fin}}(f)(U) &= \{f(x) \mid x \in U\} \end{aligned}$$

Recall also the functor $\mathsf{List} : \mathsf{Set} \to \mathsf{Set}$.

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$\mathcal{P}_{fin} X = \{\{\}, \{x_0\}, \{x_0, x_1\}, \ldots\}$

Recall also the functor List : Set $\rightarrow$ Set.

Exercise Show that there is a natural transformation α : List $\Rightarrow \mathcal{P}_{\text{fin}}$:

$X = \{x_0, x_1, \ldots\}$

- Define component morphisms for each X : Set, α_X : List $X \rightarrow \mathcal{P}_{\text{fin}}X$: $[x_0, x_1, x_0] \mapsto \{x_0, x_1\}$

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- Let's take $\alpha_X : [x_1, ..., x_n] \mapsto \{x_1, ..., x_n\}$
- The components are natural: for any function $f : X \to Y$ in Set, we have:

$$\begin{tikzcd}
\mathsf{List}\ X \arrow[r, "\alpha_X"'] \arrow[d, "\mathsf{List}\ f"'] & P_{fin} X \arrow[d, "P_{fin} f"] \\
\mathsf{List}\ Y \arrow[r, "\alpha_Y"'] & P_{fin} Y
\end{tikzcd}$$

$$P_{fin} f \circ \alpha_X = \alpha_Y \circ \mathsf{List}\ f$$

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- Let's take $\alpha_X : [x_1, ..., x_n] \mapsto \{x_1, ..., x_n\}$
- The components are natural: for any function $f : X \to Y$ in Set, we have:

$$\begin{aligned}
(\mathcal{P}_{\mathrm{fin}}(f) \circ \alpha_X)([x_1, ..., x_n]) &= \mathcal{P}_{\mathrm{fin}}(f)(\{x_1, ..., x_n\}) \\
&= \{f(x_1), ..., f(x_n)\} \\
= (\alpha_Y \circ \mathrm{List}(f))([x_1, ..., x_n]) &= \alpha_Y([f(x_1), ..., f(x_n)]) \\
&= \{f(x_1), ..., f(x_n)\}
\end{aligned}$$

A degenerate natural transformation

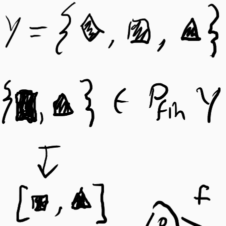
Exercise Are there any natural transformations $\beta : \mathcal{P}_{\text{fin}} \Rightarrow \text{List}$?

$[\{x_0, x_1\} \subseteq X] \in P_{\text{fin}} X$

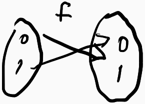
$\downarrow$

$[x_0, x_1]$ $\{0,1\} \in P_{\text{fin}} X$

$$\begin{tikzcd}
\mathcal{P}_{\text{fin}} X \arrow[r, "\beta_X"] \arrow[d, "\mathcal{P}_{\text{fin}} f"] & \text{List} X \arrow[d, "\text{List} f"] \\
\mathcal{P}_{\text{fin}} X \arrow[r, "\beta_X"] & \text{List} X
\end{tikzcd}$$



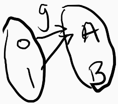
Consider $X = \{0, 1\}$, $f : X \to X : 0 \mapsto 1, 1 \mapsto 0$.



$\text{List} f \circ \beta_X = \beta_X \circ P_{\text{fin}} f$ $\beta_X : U \mapsto [\,]$

$(\beta_X \circ P_{\text{fin}} f)(\{0,1\}) = \beta_X(\{0,1\}) = [\,]$ $\{0,1\} = \{1,0\}$

$\text{List} f \circ \beta_X(\{0,1\}) = \text{List} f([\,]) = [\,]$ $P_{\text{fin}} f$



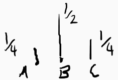
$P_{\text{fin}} g \{0,1\} = \{A\}$

An unnatural transformation[1]

Consider the discrete probability distribution functor $\mathcal{D} : \mathsf{Set} \to \mathsf{Set}$ defined as follows:

$\mathrm{supp}(\phi) = \{x \in X, x \neq 0\}$

$X = \{A, B, C\}$



$$\mathcal{D}(X) = \{\phi : X \to [0,1] \mid \mathrm{supp}(\phi) \text{ is finite and } \sum_x \phi(x) = 1\}$$

$$\mathcal{D}(f)(\phi)(y) = \sum_{x \in f^{-1}(y)} \phi(x) = \sum \{\phi(x) \mid x \in \mathrm{supp}(\phi) \text{ with } f(x) = y\}$$

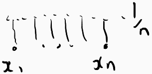
$f : X \to X$

$A \mapsto A$
$B \mapsto A$
$C \mapsto C$

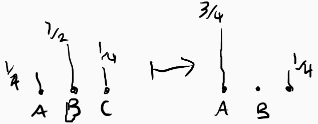
$D(f) : D(X) \to D(X)$

$D(f)(\phi) \mapsto \phi'$

Then there is an 'obvious' way to define a family of morphisms $\nu : \mathcal{P}_{\text{fin}} \Rightarrow \mathcal{D}$ as follows:



$$\nu_X : \mathcal{P}_{\text{fin}} X \to \mathcal{D}X : \{x_1, \ldots, x_n\} \mapsto \left(\phi(-) \mapsto \frac{1}{n}\right)$$



[1] Jacobs, B. Introduction to Coalgebra, §4.1

An unnatural transformation cont.

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We need to check the naturality condition.

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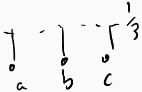
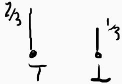


We need to check the naturality condition. Consider for example, the function $f : \{a, b, c\} \to \{\top, \bot\} : a \mapsto \top, b \mapsto \top, c \mapsto \bot$. Then

$A \longrightarrow 2$ $\qquad$ $\mathcal{D}(f) : \mathcal{D}(\{a, b, c\}) \longrightarrow \mathcal{D}(\{\top, \bot\})$

$$(\mathcal{D}(f) \circ \nu_A)(\{a, b, c\}) = \mathcal{D}(f)(\underline{\phi(-) \mapsto \tfrac{1}{3}}) =$$



$$\begin{aligned}(\nu_2 \circ \mathcal{P}_{\text{fin}}(f))(\{a, b, c\}) &= \nu_2(\{f(a), f(b), f(c)\}) \\ &= \nu_2(\{\top, \bot\}) \\ &= \left(\phi(-) \mapsto \frac{1}{2}\right)\end{aligned}$$

$\{\top, \top, \bot\}$
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Equivalence of categories: motivation

Recall that given a monoid $(M, \cdot)$ we can form a category $C(M)$ with one object, say M, and whose morphisms $M \to M$ are elements of M and with composition given by $\cdot$.

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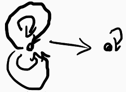

Then for every homomorphism of monoids h, we have a functor $H : C(M) \to C(N)$ that maps $\mathcal{M}$ to $\mathcal{N}$, and a morphism $m : \mathcal{M} \to \mathcal{M}$ to the arrow $h(m) : \mathcal{N} \to \mathcal{N}$.

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Now C itself defines a functor, $C : \mathrm{Mon} \to \mathrm{Ooc}$ (check identities and composition).

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$$\mathcal{C} = \left(A, \{A \xrightarrow{m} A, A \xrightarrow{m'} A, \dots\}\right)$$

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$$(CU)\mathcal{C} = (\mathrm{Mor}(\mathcal{C}), \dots)$$

Equivalence of categories

$$\begin{aligned} U : \mathrm{Ooc} &\to \mathrm{Mon} \\ \mathscr{C} &\mapsto (\mathrm{Mor}(\mathscr{C}), \circ_{\mathscr{C}}) \\ (H : \mathscr{C} \to \mathscr{D}) &\mapsto h(m) := H(m) \end{aligned}$$

$$\begin{aligned} C : \mathrm{Mon} &\to \mathrm{Ooc} \\ (M, \cdot) &\mapsto (\mathrm{Obj} = M, \mathrm{Mor} = M, \cdot) \\ (h : M \to N) &\mapsto H(M) := N \\ &\quad\; H(m : M \to M) := f(m) \end{aligned}$$

Exercise Show that $\mathrm{Ooc} \simeq \mathrm{Mon}$.

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We have that $U \circ C = \mathsf{Id}_{\mathsf{Mon}}$, so the identity morphisms give the components of a natural isomorphism $\eta : \mathsf{Id}_{\mathscr{C}} \Rightarrow U \circ C$.

Equivalence of categories

Reference:
Barr, Wells Cat theory for C.S.

$$U : \mathrm{Ooc} \to \mathrm{Mon}$$
$$\mathscr{C} \mapsto (\mathrm{Mor}(\mathscr{C}), \circ_{\mathscr{C}})$$
$$(H : \mathscr{C} \to \mathscr{D}) \mapsto h(m) := H(m)$$

$$C : \mathrm{Mon} \to \mathrm{Ooc}$$
$$(M, \cdot) \mapsto (\mathrm{Obj} = M, \mathrm{Mor} = M, \cdot)$$
$$(h : M \to N) \mapsto H(M) := N$$

H is a functor

$$H(m : M \to M) := f(m)$$

Exercise Show that $\mathrm{Ooc} \simeq \mathrm{Mon}$. $C \circ U \neq \mathrm{Id}_{\mathrm{Ooc}}$

We have that $U \circ C = \mathrm{Id}_{\mathrm{Mon}}$, so the identity morphisms give the components of a natural isomorphism $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow U \circ C$.

For ϵ we need components $\epsilon_{\mathscr{C}} : (C \circ U)\mathscr{C} \to \mathrm{Id}_{\mathrm{Ooc}}\mathscr{C}$ that are isomorphisms. If we write $\mathbb{C}$ for the single object of $\mathscr{C}$, we can just define [Note: $\epsilon_{\mathscr{C}}$ are functors, the one object of $(C \circ U)(\mathscr{C})$ is $\mathrm{Mor}(\mathscr{C})$]

$$\epsilon_{\mathscr{C}}(\mathrm{Mor}(\mathscr{C})) \mapsto \mathbb{C}$$
$$\epsilon_{\mathscr{C}}(f : \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{C})) \mapsto f : \mathbb{C} \to \mathbb{C} \in \mathrm{Mor}(\mathscr{C})$$

These component functors just rename the one object: Check that they are isos!

Composing natural transformations I: Vertical composition

Consider functors $F, G, H : \mathscr{A} \to \mathscr{B}$, and natural transformations $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$ (as on the left below).

$$
\begin{tikzcd}[column sep=huge]
\mathscr{A} \arrow[r, bend left=50, "F", ""{name=F, below}] \arrow[r, "G" description, ""{name=Ga, above}, ""{name=Gb, below}] \arrow[r, bend right=50, "H"', ""{name=H, above}] & \mathscr{B}
\arrow[Rightarrow, from=F, to=Ga, "\alpha"'] \arrow[Rightarrow, from=Gb, to=H, "\beta"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=huge]
\mathscr{A} \arrow[r, bend left=50, "F", ""{name=F, below}] \arrow[r, bend right=50, "H"', ""{name=H, above}] & \mathscr{B}
\arrow[Rightarrow, from=F, to=H, "\beta\circ\alpha"]
\end{tikzcd}
$$

Exercise Show that components $(\beta \circ \alpha)_X : FX \to HX := \beta_X \circ \alpha_X$ form a new natural transformation $\beta \circ \alpha : F \Rightarrow H$.

Composing natural transformations I: Vertical composition

$$\begin{tikzcd}[column sep=huge] \mathscr{A} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] & \mathscr{B} \arrow[Rightarrow, from=F, to=G, "\alpha"'] \arrow[Rightarrow, from=G, to=H, "\beta"'] \end{tikzcd} \qquad \begin{tikzcd}[column sep=huge] \mathscr{A} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, bend right=60, "H"'{name=H}] & \mathscr{B} \arrow[Rightarrow, from=F, to=H, "\beta\circ\alpha"] \end{tikzcd}$$

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\arrow[from=F, to=G, Rightarrow, "\alpha"']
\arrow[from=G, to=H, Rightarrow, "\beta"']
\end{tikzcd}
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\mathscr{A} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, bend right=60, "H"'{name=H}] & \mathscr{B}
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FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[rr, bend left] \arrow[dr, phantom, "="] & GX \arrow[r, "\beta_X"] \arrow[d, "Gf"'] \arrow[dr, phantom, "="] & HX \arrow[d, "Hf"] \\
FY \arrow[r, "\alpha_Y"'] \arrow[rr, bend right] & GY \arrow[r, "\beta_Y"'] & HY
\end{tikzcd}$$

Want to show

$$Hf \circ (\beta \circ \alpha)_X = (\beta \circ \alpha)_Y \circ Ff$$

$$Hf \circ \beta_X \circ \alpha_X = \beta_Y \circ \alpha_Y \circ Ff$$

$$(Hf \circ \beta_X) \circ \alpha_X = (\beta_Y \circ Gf) \circ \alpha_X$$

$$= \beta_Y \circ (Gf \circ \alpha_X) = \beta_Y \circ \alpha_Y \circ Ff$$

$$= (\beta \circ \alpha)_Y \circ Ff$$

Composing natural transformations II: Horizontal composition

Given the four functors and two natural transformations on the left below:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{C} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "I\circ G"'] \arrow[r, phantom, "\Downarrow\beta\star\alpha"] & \mathscr{C} \end{tikzcd}$$

Exercise Show that the components $(\beta \star \alpha)_X = \beta_{GX} \circ H(\alpha_X) = I(\alpha_X) \circ \beta_{FX}$ define a new natural transformation $\beta \star \alpha : H \circ F \Rightarrow I \circ G$

$$HFX \xrightarrow[(\beta\star\alpha)_X]{} IGX$$

$$H(\alpha_X) : HFX \to HGX$$

$$HGX \to IGX$$

Composing natural transformations II: Horizontal composition

Given the four functors and two natural transformations on the left below:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{C} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "I\circ G"'] \arrow[r, phantom, "\Downarrow\beta\star\alpha"] & \mathscr{C} \end{tikzcd}$$

Exercise Show that the components $(\beta \star \alpha)_X := \underline{\beta_{GX} \circ H(\alpha_X)} = I(\alpha_X) \circ \beta_{FX}$ define a new natural transformation $\beta \star \alpha : H \circ F \Rightarrow I \circ G$

$$\begin{tikzcd} HFA \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{FA}"'] & HGA \arrow[d, "\beta_{GA}"] \\ IFA \arrow[r, "I(\alpha_A)"'] & IGA \end{tikzcd} \qquad \begin{tikzcd} HFA \arrow[r, "H(\alpha_A)"] \arrow[d, "HFf"'] \arrow[dr, phantom, "="] & HGA \arrow[r, "\beta_{GA}"] \arrow[d, "H(Gf)"] \arrow[dr, phantom, "="] & IGA \arrow[d, "I(Gf)"] \\ HFA' \arrow[r, "H(\alpha_{A'})"'] & HGA' \arrow[r, "\beta_{GA'}"'] & IGA' \end{tikzcd}$$

Lemma

functors preserve commutative diagrams

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "h"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "i"'] & \cdot \end{tikzcd} \xrightarrow{\quad F \quad} \begin{tikzcd} \cdot \arrow[r, "F(f)"] \arrow[d, "F(h)"'] & \cdot \arrow[d, "F(g)"] \\ \cdot \arrow[r, "F(i)"'] & \cdot \end{tikzcd}$$

$$g \circ f = i \circ h$$

$$F(g \circ f) = F(g) \circ F(f)$$
$$= F(i \circ h) = F(i) \circ F(h)$$