Universal Constructions

ITI9200, Spring 2021

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A *universal construction* is a description of a construction within a category that determines it uniquely up to a canonical isomorphism. This is the best kind of description we can hope for in a categorical setting, where we do not have direct access to the internal structure of the objects we are working with.

Universal constructions are characterized by *universal properties*, which assert that the construction itself has some property, and that if any other construction of the same "shape" in the category has that property then there is a canonical relationship between the two.

In this section we will introduce several universal constructions. We will do this in a deliberately methodical way, in order to emphasize their similarities.

1 Terminal and Initial Objects

1.1 Terminal Objects

In the category Set, a singleton set S has the property that given any set X there is a unique function from X to S, namely, the constant function on the only element of S. This is a behavioral characterization that we may state in an arbitrary category.

Definition 1.1 (terminal object)
In any category, a *terminal object* is an object T with the property that for any object X there is a unique morphism $x : \mathrm{X} \to \mathrm{T}$.

We write "$!\,\mathrm{X}$" for the unique map from an object X to a terminal object and refer to it as a *bang map*.

Whenever some construction has a certain relationship to all constructions of the same shape within a category, it must, in particular, have this relationship to itself. Socrates' dictum to "know thyself" is as important in category theory as it is in life. So whenever we encounter a universal construction we will see what we can learn about it by "probing it with itself". In the case of a terminal object, this means choosing $\mathrm{X} := \mathrm{T}$ in the definition.

Lemma 1.2 (identity expansion for terminals)
If T is a terminal object then $!\,\mathrm{T} = \mathrm{id}\,\mathrm{T}$.

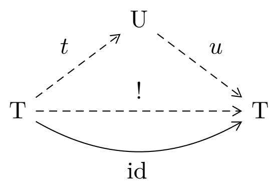
Proof. By assumption, $!\,\mathrm{T}$ is the unique map $t : \mathrm{T} \to \mathrm{T}$, but $\mathrm{id}\,\mathrm{T}$ is an arrow in the same hom set. □

Universal constructions are each unique up to a unique structure-preserving isomorphism. In the case of a terminal object, the structure to be preserved is trivial: it's just a single object. Consequently, we obtain an especially strong uniqueness property.

Lemma 1.3 (uniqueness of terminals)
When they exist, terminal objects are unique up to a unique isomorphism.

Proof. Suppose that T and U are two terminal objects in a category. Because U is terminal there is a unique arrow $t : \mathrm{T} \to \mathrm{U}$. Likewise, because T is terminal there is a unique arrow $u : \mathrm{U} \to \mathrm{T}$. Then we have:

$$
\begin{array}{ll}
 & t \cdot u : \mathrm{T} \to \mathrm{T} \\
= & [\text{T is terminal}] \\
 & !\,\mathrm{T} : \mathrm{T} \to \mathrm{T} \\
= & [\text{identity expansion for terminals}] \\
 & \mathrm{id}\,\mathrm{T} : \mathrm{T} \to \mathrm{T}
\end{array}
$$



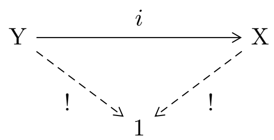
Symmetrically, we have that $u \cdot t = \mathrm{id}\,\mathrm{U}$. So t is an isomorphism. By the universal property of the terminal object U, the hom set $\mathrm{T} \to \mathrm{U}$ is a singleton, so it must be the only one. □

Because terminal objects are unique up to unique isomorphism, we write "1" to refer to an arbitrary terminal object of a category.

Exercise 1 (pre-composing with a bang)
Use the universal property of a terminal object to prove the following:

If 1 is a terminal object, then for any arrow $i : \mathrm{Y} \to \mathrm{X}$ we have,

$$i \cdot !\,\mathrm{X} \quad = \quad !\,\mathrm{Y} \quad : \quad \mathrm{Y} \to 1$$



As mentioned, in Set, any singleton set is terminal. Likewise, in Cat, any singleton category is. In Mon, the trivial monoid (having only the identity element) is terminal.

Exercise 2
Work out what a terminal object is in the category of preordered sets. Then determine when a preordered set regarded as a category itself has a terminal object.

1.2 Global Elements

In Set, there is a bijection between the elements of a set X and the functions from a singleton set to X that associates to each $x \in \mathrm{X}$ the function $\ulcorner x \urcorner : 1 \to \mathrm{X}$ mapping $\star \longmapsto x$. We can use this behavioral characterization to define a categorical analogue for set membership.

Definition 1.4 (global element)
In a category with a terminal object, a *global element* or *point* of an object A is an element of the hom set $1 \to \mathrm{A}$.

Definition 1.5 (generalized element)
In contrast, a *generalized element* of an object A is just a morphism with codomain A; in other words, an object of the slice category over A.

In Set, we can determine whether or not two functions are the same by probing them with points because two parallel functions f, g : Set $(\mathrm{X} \to \mathrm{Y})$ are equal just in case they agree on all points:

$$f = g \quad := \quad \forall\, x \in \mathrm{X} \,.\, f(x) = g(x)$$

This is known as the principle of *function extensionality.* Here is a categorical analogue:

Definition 1.6 (well-pointed category)
A category with a terminal object is *well-pointed* if for every parallel pair of arrows $f, g : \mathrm{A} \to \mathrm{B}$, and global element of their shared domain $a : 1 \to \mathrm{A}$,

$$a \cdot f = a \cdot g \qquad \text{implies} \qquad f = g$$

Notice the similarity to the definition of an epimorphism. In fact, we say that a category is well-pointed if its points are *jointly epic*, that is, if points are collectively able to distinguish arrows.

Exercise 3
In contrast to the case with global elements, in any category we can determine whether or not two parallel arrows are the same by probing them with generalized elements. Prove this. (*Hint*: for any pair of arrows, a single "probe" suffices.)

1.3 Initial Objects

The concept dual to that of a terminal object is of an initial object.

Definition 1.7 (initial object)
In any category, an *initial object* is an object S with the property that for any object X there is a unique morphism $x : \mathrm{S} \to \mathrm{X}$.

We write "¡ X" for the unique map in $\mathrm{S} \to \mathrm{X}$ and refer to it as a *cobang map.* By probing an initial object with itself we obtain a result dual to lemma 1.2:

Lemma 1.8 (identity expansion for initials)
If S is an initial object then ¡ S = id S.

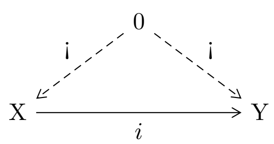
Exercise 4 (uniqueness of initials)
Check that when they exist, initial objects are unique up to a unique isomorphism.

We write "0" to refer to an arbitrary initial object of a category.

Lemma 1.9 (post-composing with a cobang)
dual to exercise 1:

If 0 is an initial object, then for any arrow $i : \mathrm{X} \to \mathrm{Y}$ we have,

$$¡\,\mathrm{X} \cdot i \quad = \quad ¡\,\mathrm{Y} \quad : \quad 0 \to \mathrm{Y}$$



In Set, the empty set is initial. Likewise, in Cat, the empty category is. In Mon, the trivial monoid is initial as well as terminal. (An object which is both terminal and initial is known as a *null object.*)

Exercise 5
Dualize exercise 2 by working out what an initial object is in the category of preordered sets, and determine when a preordered set regarded as a category has an initial object.

2 Products

2.1 Products of Objects

In the category SET the set of ordered pairs:

$$\mathrm{A} \times \mathrm{B} \quad := \quad \{(a\,, b) \mid a \in \mathrm{A} \text{ and } b \in \mathrm{B}\}$$

comes equipped with two projection functions,

$$\begin{array}{rcl} \mathrm{A} \times \mathrm{B} & \xrightarrow{\pi_0} & \mathrm{A} \\ (a\,, b) & \longmapsto & a \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{A} \times \mathrm{B} & \xrightarrow{\pi_1} & \mathrm{B} \\ (a\,, b) & \longmapsto & b \end{array}$$

such that for any ordered pair $x \in \mathrm{A} \times \mathrm{B}$,

$$x \;=\; (\pi_0\, x\,, \pi_1\, x)$$

So having a pair of elements, one from the set A and one from the set B, is the same thing as having a single element of the set $\mathrm{A} \times \mathrm{B}$:

- given an $a \in \mathrm{A}$ and $b \in \mathrm{B}$ we make an element of $\mathrm{A} \times \mathrm{B}$ by forming the tuple $(a\,, b)$,
- and given an element $x \in \mathrm{A} \times \mathrm{B}$ we recover elements of A and B by taking the projections.

Not every category is well-pointed like SET is, or for that matter, even has a terminal object. So to describe this situation categorically we must use generalized elements. This motivates the definition of products in an arbitrary category.

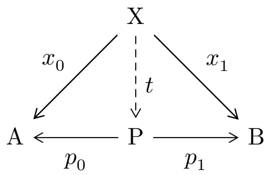
Definition 2.1 (product of objects)
In any category, a cartesian[1] *product* of objects A and B is a span on A and B,

$$\mathrm{A} \xleftarrow{p_0} \mathrm{P} \xrightarrow{p_1} \mathrm{B}$$

with the property that for any span on A and B,

$$\mathrm{A} \xleftarrow{x_0} \mathrm{X} \xrightarrow{x_1} \mathrm{B}$$

there is a unique map $t : \mathrm{X} \to \mathrm{P}$ such that $t \cdot p_0 = x_0$ and $t \cdot p_1 = x_1$:



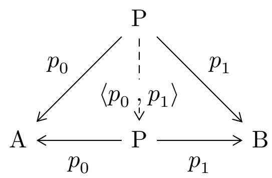
This says that there is a bijection between ordered pairs of maps $(x_0\,, x_1)$ and single maps t such that the diagram commutes. We call A and B the *factors* of the product, p_0 and p_1 its (coordinate) *projections* and t the *tuple* of x_0 and x_1 and write it as "$\langle x_0\,, x_1\rangle$".

Let's see what we can learn from probing a product with itself by choosing $\mathrm{X} := \mathrm{P}$ and $(x_0\,, x_1) := (p_0\,, p_1)$.

[1] The word "cartesian" is sometimes used for emphasis to distinguish this construction from various other categorical constructions also known as "products", for example, the *monoidal* product, $- \otimes -$ that we will meet later in the course.

Lemma 2.2 (identity expansion for products)
If P is a product of A and B with projections p_0 and p_1, then $\langle p_0 \, , p_1 \rangle = \mathrm{id}\,\mathrm{P}$.

Proof. By assumption, $\langle p_0 \, , p_1 \rangle$ is the unique map $t : \mathrm{P} \to \mathrm{P}$ with the property that, $t \cdot p_0 = p_0$ and $t \cdot p_1 = p_1$:



but by the left unit law of composition, $\mathrm{id}\,\mathrm{P}$ has this property. ☐

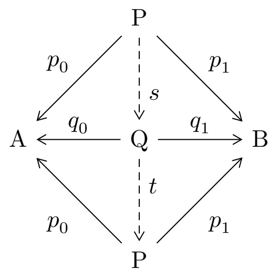
Because products are structures characterized by a universal property, we expect them to be uniquely determined up to a unique structure-preserving isomorphism. This is indeed the case:

Lemma 2.3 (uniqueness of products)
When they exist, products of objects are unique up to a unique projection-preserving isomorphism.

Proof. Suppose that the spans:

$$\mathrm{A} \xleftarrow{p_0} \mathrm{P} \xrightarrow{p_1} \mathrm{B} \qquad \text{and} \qquad \mathrm{A} \xleftarrow{q_0} \mathrm{Q} \xrightarrow{q_1} \mathrm{B}$$

are both products of A and B. Because Q is a product there is a unique $s : \mathrm{P} \to \mathrm{Q}$ such that $s \cdot q_0 = p_0$ and $s \cdot q_1 = p_1$. Likewise, because P is a product there is a unique $t : \mathrm{Q} \to \mathrm{P}$ such that $t \cdot p_0 = q_0$ and $t \cdot p_1 = q_1$.



Then for $i \in \{0, 1\}$:

$$\begin{array}{cl} & s \cdot t \cdot p_i \\ = & [\text{P is a product}] \\ & s \cdot q_i \\ = & [\text{Q is a product}] \\ & p_i \end{array}$$

By the universal property of the product P we get $s \cdot t = \langle p_0 \, , p_1 \rangle : \mathrm{P} \to \mathrm{P}$. And by identity expansion for products, $s \cdot t = \mathrm{id}\,\mathrm{P}$. Reversing the roles of P and Q, we get that $t \cdot s = \mathrm{id}\,\mathrm{Q}$ as well. So s is an isomorphism. By the universal property of the product Q, it is the only one that respects the coordinate projections. ☐

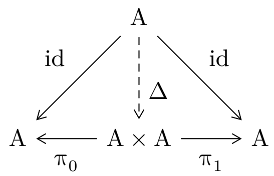
Because products are determined as uniquely as is possible by a behavioral characterization, we write "$\mathrm{A} \times \mathrm{B}$" to refer to an arbitrary product of A and B. When the product in question is clear from context, we refer to the two coordinate projections generically as "π_0" and "π_1".

In the category SET, the set of ordered pairs is a cartesian product. Likewise, in CAT, the ordered pair category is. This justifies the notation "$- \times -$" that we used in both cases.

Note that unlike the case with terminal objects, there is not necessarily a unique isomorphism between two products of the same factors. For example, in SET the identity function, $(x \, , y) \longmapsto (x \, , y)$, and swap map, $(x \, , y) \longmapsto (y \, , x)$, are both isomorphisms $\mathrm{A} \times \mathrm{A} \to \mathrm{A} \times \mathrm{A}$. But only the former respects the coordinate projections.

Definition 2.4 (diagonal map)
For every object A, the universal property of the product gives a canonical *diagonal map*, which duplicates its argument:

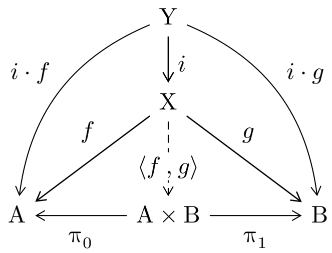
$$\Delta\mathrm{A} \quad := \quad \langle \mathrm{id}\,\mathrm{A}\,,\,\mathrm{id}\,\mathrm{A}\rangle \quad : \quad \mathrm{A} \to \mathrm{A}\times\mathrm{A}$$



Exercise 6 (pre-composing with a tuple)
Use the diagram below and the universal property of a product of objects to prove the following:

For a product $\mathrm{A}\times\mathrm{B}$, a tuple $\langle f\,,\,g\rangle : \mathrm{X}\to\mathrm{A}\times\mathrm{B}$, and an arrow $i : \mathrm{Y}\to\mathrm{X}$,

$$i\cdot\langle f\,,\,g\rangle \quad = \quad \langle i\cdot f\,,\,i\cdot g\rangle \quad : \quad \mathrm{Y}\to\mathrm{A}\times\mathrm{B}$$



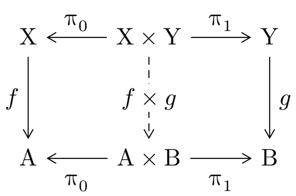
2.2 Product Functors

We can use the universal property of a product of objects to define a product of arrows as well:

Definition 2.5 (product of arrows)
Given products of objects $\mathrm{X}\times\mathrm{Y}$ and $\mathrm{A}\times\mathrm{B}$, and arrows between their respective factors $f : \mathrm{X}\to\mathrm{A}$ and $g : \mathrm{Y}\to\mathrm{B}$, we define the *product of arrows* by:

$$\begin{array}{lcl} f\times g & : & \mathrm{X}\times\mathrm{Y}\to\mathrm{A}\times\mathrm{B} \\ f\times g & := & \langle \pi_0\cdot f\,,\,\pi_1\cdot g\rangle \end{array}$$

By the universal property of its codomain product the tuple arrow $f\times g$ is the unique morphism making the two squares commute:



This allows us to characterize the product as a functor:

Lemma 2.6 (functoriality of products)
If a category $\mathbb{C}$ has products for each pair of objects, then the given definition of products for arrows yields a functor,

$$\begin{array}{ccc} & \underline{-\times-} & \\ \mathbb{C}\times\mathbb{C} & \longrightarrow & \mathbb{C} \\ (\mathrm{A}\,,\,\mathrm{B}) & \longmapsto & \mathrm{A}\times\mathrm{B} \\ (f\,,\,g) & \longmapsto & f\times g \end{array}$$

called the *product functor*.

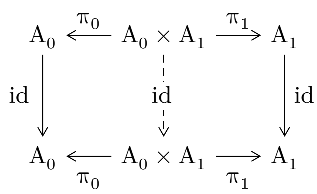
Before giving the proof, we pause to explain this statement, as it is easy to be confused about what is being asserted. In the lemma, "$\mathbb{C} \times \mathbb{C}$" is the ordered pair category; i.e. the product of $\mathbb{C}$ with itself in Cat. In contrast, "$- \times -$" is the name of an alleged functor having as domain the category $\mathbb{C} \times \mathbb{C}$ and as codomain the category $\mathbb{C}$. This alleged functor takes an ordered pair of $\mathbb{C}$-objects $(\mathrm{A}\,,\mathrm{B}) : \mathbb{C}_0 \times \mathbb{C}_0$ to the $\mathbb{C}$-object that is their cartesian product $\mathrm{A} \times \mathrm{B} : \mathbb{C}_0$; and takes an ordered pair of $\mathbb{C}$-arrows $(f\,,g) : \mathbb{C}_1 \times \mathbb{C}_1$ to the $\mathbb{C}$-arrow that is their cartesian product $f \times g : \mathbb{C}_1$ as just defined.

Proof. In order to prove that $- \times -$ is indeed a functor, we must show that it preserves the composition structure.

nullary composition: We must show that

$$\mathrm{id}\,\mathrm{A}_0 \times \mathrm{id}\,\mathrm{A}_1 \;=\; \mathrm{id}\,(\mathrm{A}_0 \times \mathrm{A}_1)$$

In the diagram,

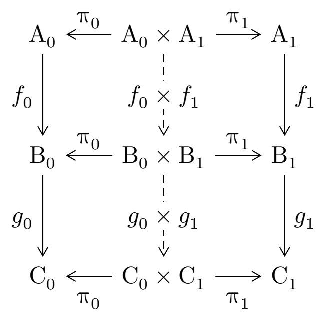


the arrow $\mathrm{id}\,(\mathrm{A}_0 \times \mathrm{A}_1)$ makes both squares commute. But by the definition of the product of arrows, $\mathrm{id}\,\mathrm{A}_0 \times \mathrm{id}\,\mathrm{A}_1$ is the unique arrow that makes both squares commute.

binary composition: We must show that

$$(f_0 \cdot g_0) \times (f_1 \cdot g_1) \;=\; (f_0 \times f_1) \cdot (g_0 \times g_1)$$

In the diagram,



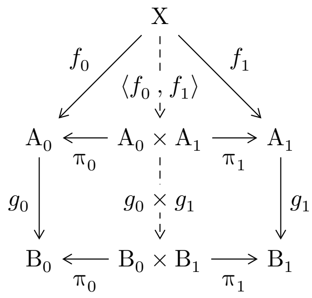
the top two squares commute by the definition of $f_0 \times f_1$ and the bottom two squares commute by the definition of $g_0 \times g_1$. By pasting, the rectangle comprising the two left squares commutes, and likewise the rectangle comprising the two right squares. By definition, $(f_0 \cdot g_0) \times (f_1 \cdot g_1)$ is the unique arrow from $\mathrm{A}_0 \times \mathrm{A}_1$ to $\mathrm{C}_0 \times \mathrm{C}_1$ making the outer square commute.

□

Exercise 7 (tuple-product composition)
Use the universal property of a product of objects to prove the following:

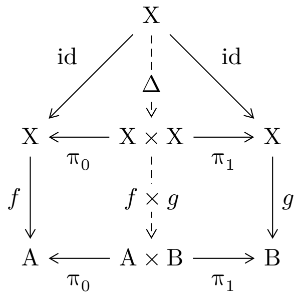
For tuple $\langle f_0\,, f_1\rangle : \mathrm{X} \to \mathrm{A}_0 \times \mathrm{A}_1$ and product $g_0 \times g_1 : \mathrm{A}_0 \times \mathrm{A}_1 \to \mathrm{B}_0 \times \mathrm{B}_1$,

$$\langle f_0\,, f_1\rangle \cdot (g_0 \times g_1) \;=\; \langle f_0 \cdot g_0\,, f_1 \cdot g_1\rangle$$



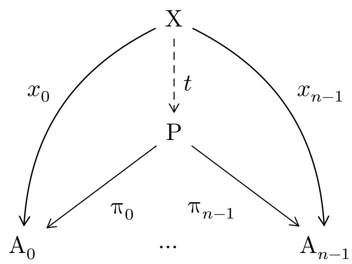
Corollary 2.7 (tuple factorization)
A tuple $\langle f , g \rangle : X \to A \times B$ factors through the diagonal map as,

$$\langle f , g \rangle \ = \ \Delta X \cdot (f \times g)$$

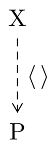


2.3 Finite Products

Returning to the theme of unbiased presentations, we would like to define an n-ary product for each $n \in \mathbb{N}$. Let's think about what the universal property of such a construction would be. A product of n factors would consist of an object P, together with a coordinate projection, $\pi_i : P \to A_i$ for each factor such that for any n-ary span $x_i : X \to A_i$ over the same factors there is a unique n-tuple map $t : X \to P$ with $t \cdot \pi_i = x_i$.



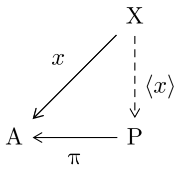
For $n := 0$, a *nullary product* is an object P (requiring no coordinate projections) such that for any object X (requiring no maps to the zero factors) there is a unique null-tuple $\langle\,\rangle : X \to P$ (satisfying no conditions):



But this is just a terminal object!

For $n := 1$, a *unary product* of an object A is an object P with a single coordinate projection, $\pi : P \to A$ such that for any arrow $x : X \to A$ there is a unique one-tuple $\langle x \rangle : X \to P$ with

$\langle x \rangle \cdot \pi = x$:



A moment's thought confirms that the choice of $P := A$ and $\pi := \mathrm{id}\, A$ (and thus $\langle x \rangle := x$) satisfies this property. So any object is a unary product of itself.

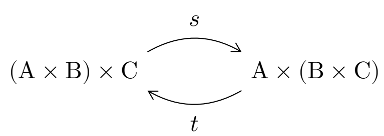
Binary products have already been defined, so we have left to consider products of three or more factors. A ternary product is an object $A \times B \times C$, equipped with three coordinate projection maps such that for any 3-legged span over its factors there is a unique map from the apex to $A \times B \times C$ commuting with the coordinate projections. But this is the same universal property enjoyed by $(A \times B) \times C$, which has projections $\pi_0 \cdot \pi_0$ to A, $\pi_0 \cdot \pi_1$ to B and π_1 to C. Any span over A, B and C contains a subspan over A and B, so by the universal property of $A \times B$, has a unique map from the apex to this product, which together with the C leg of the span gives us a unique map from the apex to $(A \times B) \times C$. The product of four or more factors is analogous.

Of course, there is nothing special about the choice of bracketing:

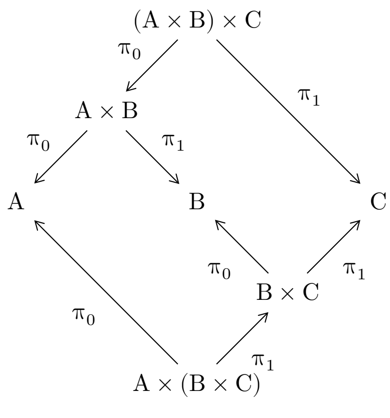
Lemma 2.8 (product associator)
Products are associative, up to isomorphism:

$$(A \times B) \times C \quad \cong \quad A \times (B \times C)$$

Proof. The maps back and forth,



become clear when we draw the diagram showing how each compound product projects to the three factors, A, B, and C:



From this we can simply read off:

$$s \;:=\; \langle \pi_0 \cdot \pi_0 \,, \langle \pi_0 \cdot \pi_1 \,, \pi_1 \rangle \rangle \;:\; (A \times B) \times C \to A \times (B \times C)$$
$$t \;:=\; \langle \langle \pi_0 \,, \pi_1 \cdot \pi_0 \rangle \,, \pi_1 \cdot \pi_1 \rangle \;:\; A \times (B \times C) \to (A \times B) \times C$$

And then we check:

$$
\begin{aligned}
& s \cdot t \\
= \; & [\text{definition } t] \\
& s \cdot \langle \langle \pi_0 \, , \pi_1 \cdot \pi_0 \rangle \, , \pi_1 \cdot \pi_1 \rangle \\
= \; & [\text{pre-composing with a tuple}] \\
& \langle \langle s \cdot \pi_0 \, , s \cdot \pi_1 \cdot \pi_0 \rangle \, , s \cdot \pi_1 \cdot \pi_1 \rangle \\
= \; & [\text{definition } s] \\
& \langle \langle \pi_0 \cdot \pi_0 \, , \pi_0 \cdot \pi_1 \rangle \, , \pi_1 \rangle \\
= \; & [\text{pre-composing with a tuple}] \\
& \langle \pi_0 \cdot \langle \pi_0 \, , \pi_1 \rangle \, , \pi_1 \rangle \\
= \; & [\text{identity expansion for products}] \\
& \langle \pi_0 \cdot \mathrm{id} \; , \pi_1 \rangle \\
= \; & [\text{composition unit law}] \\
& \langle \pi_0 \, , \pi_1 \rangle \\
= \; & [\text{identity expansion for products}] \\
& \mathrm{id}
\end{aligned}
$$

And similarly, $t \cdot s = \mathrm{id}$. □

Up to isomorphism, the cartesian product has the structure of a monoid:

Lemma 2.9 (product unitor)
A terminal object is a unit for products, up to isomorphism:

$$1 \times \mathrm{A} \quad \cong \quad \mathrm{A} \quad \cong \quad \mathrm{A} \times 1$$

Proof. The projection $\pi_0 : \mathrm{A} \times 1 \to \mathrm{A}$ is an isomorphism, with inverse $\langle \mathrm{id}\, \mathrm{A} \, , ! \, \mathrm{A} \rangle : \mathrm{A} \to \mathrm{A} \times 1$.

- By the universal property of the product,

$$\langle \mathrm{id}\, \mathrm{A} \, , ! \, \mathrm{A} \rangle \cdot \pi_0 \quad = \quad \mathrm{id}\, \mathrm{A} \quad : \quad \mathrm{A} \to \mathrm{A}$$

- Going the other way,

$$
\begin{aligned}
& \pi_0 \cdot \langle \mathrm{id}\, \mathrm{A} \, , ! \, \mathrm{A} \rangle \quad : \quad \mathrm{A} \times 1 \to \mathrm{A} \times 1 \\
= \; & [\text{pre-composing with a tuple}] \\
& \langle \pi_0 \cdot \mathrm{id}\, \mathrm{A} \, , \pi_0 \cdot ! \, \mathrm{A} \rangle \\
= \; & [\text{composition unit law and pre-composing with a bang}] \\
& \langle \pi_0 \, , ! \, (\mathrm{A} \times 1) \rangle \\
= \; & [\text{universal property of a terminal object}] \\
& \langle \pi_0 \, , \pi_1 \rangle \\
= \; & [\text{identity expansion for products}] \\
& \mathrm{id}\, (\mathrm{A} \times 1)
\end{aligned}
$$

□

And furthermore, this monoid is commutative:

Lemma 2.10 (product symmetry)
Products are symmetric, up to isomorphism:

$$\mathrm{A} \times \mathrm{B} \quad \cong \quad \mathrm{B} \times \mathrm{A}$$

Proof. The *swap map* $\sigma_{A,B} := \langle \pi_1 , \pi_0 \rangle : A \times B \to B \times A$ is an isomorphism with inverse the swap map $\sigma_{B,A} := \langle \pi_1 , \pi_0 \rangle : B \times A \to A \times B$:

$$
\begin{array}{cl}
 & \sigma_{A,B} \cdot \sigma_{B,A} \\
= & [\text{definition}] \\
 & \langle \pi_1 , \pi_0 \rangle \cdot \langle \pi_1 , \pi_0 \rangle \\
= & [\text{pre-composing with a tuple}] \\
 & \langle \langle \pi_1 , \pi_0 \rangle \cdot \pi_1 , \langle \pi_1 , \pi_0 \rangle \cdot \pi_0 \rangle \\
= & [\text{universal property of a product}] \\
 & \langle \pi_0 , \pi_1 \rangle \\
= & [\text{identity expansion for products}] \\
 & \mathrm{id}\,(A \times B)
\end{array}
$$

and symmetrically $\sigma_{B,A} \cdot \sigma_{A,B} = \mathrm{id}\,(B \times A)$ □

To have *finite products* – that is, n-ary products for all $n \in \mathbb{N}$, it suffices to have binary products and a terminal object. A category with all finite products is called a *cartesian category*.

3 Coproducts

A coproduct is the dual construction to a product. Categorically, that is all there is to say about the matter. But despite this fact the coproducts that we encounter in various categories *feel* quite different from the products, and in categories where products and coproducts occur together the symmetry is often broken by a distributive law.

First, we record for convenience, but without further comment, the duals of our main results about products. If you're new to this, it would be an excellent exercise first to go back and see why these are the respective dual theorems, and then to prove each one explicitly – that is, by actually going through the argument, rather than by just saying, "by duality, Qed".

3.1 Coproducts of Objects

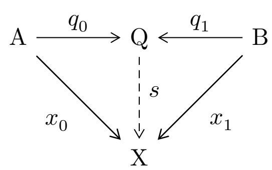
Definition 3.1 (coproduct of objects)
In any category, a *coproduct* of objects A and B is a cospan on A and B,

$$A \xrightarrow{q_0} Q \xleftarrow{q_1} B$$

with the property that for any cospan on A and B,

$$A \xrightarrow{x_0} X \xleftarrow{x_1} B$$

there is a unique map $s : Q \to X$ such that $q_0 \cdot s = x_0$ and $q_1 \cdot s = x_1$:



We call A and B the *cases* of the coproduct, q_0 and q_1 its *insertions* and s the *cotuple* of x_0 and x_1, and write it as "$[x_0 , x_1]$".

Probing a coproduct with itself by choosing $X := Q$ and $(x_0 , x_1) := (q_0 , q_1)$, we learn:

Lemma 3.2 (identity expansion for coproducts)
If Q is a coproduct of A and B with insertions q_0 and q_1, then $[q_0 , q_1] = \mathrm{id}\,Q$.

And being characterized by a universal property, we expect:

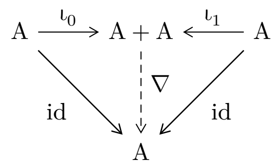
Lemma 3.3 (uniqueness of coproducts)
When they exist, coproducts of objects are unique up to a unique insertion-preserving isomorphism.

We write "A + B" to refer to an arbitrary coproduct of A and B, When the coproduct in question is clear from context, we refer to the two case insertions generically as "ι_0" and "ι_1".

In the category SET, the disjoint union of two sets is their coproduct. In CAT, there is something similar: $\mathbb{C} + \mathbb{D}$ is the category whose collection of objects is the disjoint union of those of $\mathbb{C}$ and $\mathbb{D}$ and whose homs between pairs of $\mathbb{C}$-objects is the same as in $\mathbb{C}$, and likewise for $\mathbb{D}$, but where the "heteromorphism" sets between objects originating in different categories are empty.

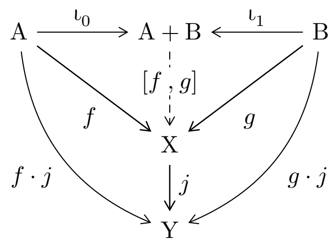
Definition 3.4 (codiagonal map)
For every object A, the universal property of the coproduct gives a canonical *codiagonal map*, which forgets about case distinction:

$$\nabla \mathrm{A} \quad := \quad [\mathrm{id}\,\mathrm{A}\ ,\ \mathrm{id}\,\mathrm{A}] \quad : \quad \mathrm{A} + \mathrm{A} \to \mathrm{A}$$



Lemma 3.5 (*post-composing with a cotuple*)
For a coproduct A + B, a cotuple $[f\ ,\ g] : \mathrm{A} + \mathrm{B} \to \mathrm{X}$, and an arrow $j : \mathrm{X} \to \mathrm{Y}$,

$$[f\ ,\ g] \cdot j \ = \ [f \cdot j\ ,\ g \cdot j] \ : \ \mathrm{A} + \mathrm{B} \to \mathrm{Y}$$

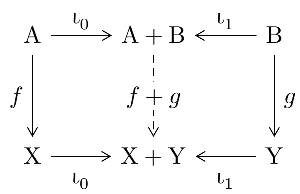


3.2 Coproduct Functors

Definition 3.6 (coproduct of arrows)
Given coproducts of objects A+B and X+Y, and arrows between their respective cases $f : \mathrm{A} \to \mathrm{X}$ and $g : \mathrm{B} \to \mathrm{Y}$, we define the *coproduct of arrows* by:

$$\begin{array}{lcl} f + g & : & \mathrm{A} + \mathrm{B} \to \mathrm{X} + \mathrm{Y} \\ f + g & := & [f \cdot \iota_0\ ,\ g \cdot \iota_1] \end{array}$$

By the universal property of its domain coproduct the cotuple arrow $f + g$ is the unique morphism making the two squares commute:



This allows us to characterize the coproduct as a functor:

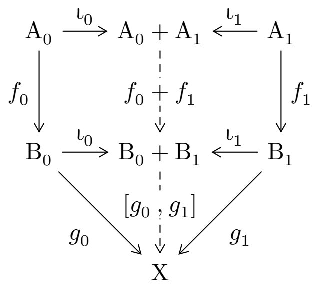
Lemma 3.7 (functoriality of coproducts)
If a category $\mathbb{C}$ has coproducts for each pair of objects, then the given definition of coproducts for arrows yields a functor,

$$\begin{array}{ccc} & \underline{-+-} & \\ \mathbb{C} \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ (\mathrm{A}\,,\mathrm{B}) & \longmapsto & \mathrm{A}+\mathrm{B} \\ (f\,,g) & \longmapsto & f+g \end{array}$$

called the *coproduct functor*.

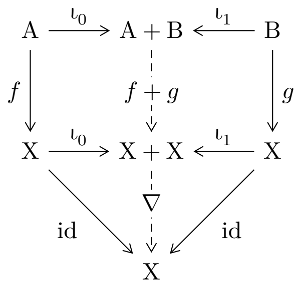
Lemma 3.8 (coproduct-cotuple composition)
For coproduct $f_0 + f_1 : \mathrm{A}_0 + \mathrm{A}_1 \to \mathrm{B}_0 + \mathrm{B}_1$ and cotuple $[g_0\,, g_1] : \mathrm{B}_0 + \mathrm{B}_1 \to \mathrm{X}$,

$$(f_0 + f_1) \cdot [g_0\,, g_1] \;=\; [f_0 \cdot g_0\,, f_1 \cdot g_1]$$



Corollary 3.9 (cotuple factorization)
A cotuple $[f\,, g] : \mathrm{A} + \mathrm{B} \to \mathrm{X}$ factors through the codiagonal map as,

$$[f\,, g] \;=\; (f + g) \cdot \nabla\mathrm{X}$$



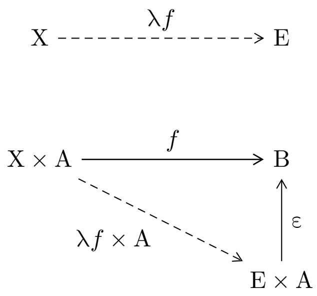
4 Exponentials

As functional programmers, we are familiar with the idea of function *currying*, that is, of viewing a function of two arguments as a *higher-order function* that takes the first argument and returns a new function, which, when provided the second argument, computes the same result as the original function does when given both arguments at once. Once we get used to working with high-order functions, we wonder how we ever managed to program any other way.

4.1 Exponentials of Objects

Exponential objects are the categorical analogue of set-theoretic function space, allowing us to characterize function currying and λ-abstraction.

Definition 4.1 (exponential object)
In a category with binary products, an *exponential* of objects A and B is an object E together with an arrow $\varepsilon : E \times A \to B$ with the property that for any object X and arrow $f : X \times A \to B$ there is a unique arrow $\lambda f : X \to E$ such that $(\lambda f \times A) \cdot \varepsilon = f$:



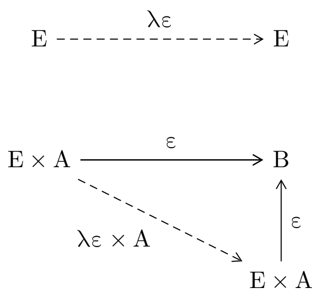
We call ε the *evaluation map* of the exponential, and λf the *exponential transpose* or "curry" of f.

Notice that the "such that" clause of the definition lets us recover f from λf: just take the product with $\mathrm{id}\,A$ and compose with ε. This is just "uncurrying" to functional programmers. If this isn't clear to you then go back to the definitions of product and coproduct and see how the same principle allows us to recover f and g from $\langle f , g \rangle$ and from $[f , g]$, respectively.

Let's see what we learn from probing an exponential with itself by choosing $X := E$ and $f := \varepsilon$.

Lemma 4.2 (identity expansion for exponentials)
If E is an exponential of A and B then $\lambda\varepsilon = \mathrm{id}\,E$.

Proof. By assumption, $\lambda\varepsilon$ is the unique map $t : E \to E$ with the property that $(t \times \mathrm{id}\,A) \cdot \varepsilon = \varepsilon$:



but $(\mathrm{id}\,E \times \mathrm{id}\,A) \cdot \varepsilon = \mathrm{id}\,(E \times A) \cdot \varepsilon = \varepsilon$, so $\mathrm{id}\,E$ has this property. ☐

To summarize:

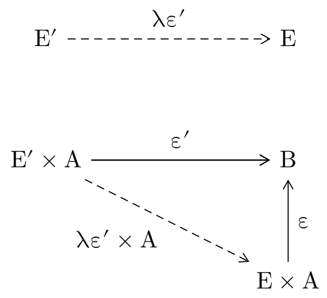
- currying the evaluation map yields the identity on the exponential, and
- uncurrying the identity on the exponential yields the evaluation map.

Because exponentials are structures characterized by a universal property, we expect them to be unique up to a unique structure-preserving isomorphism. This should be familiar by now.

Lemma 4.3 (uniqueness of expoentials)
When they exist, exponentials are unique up to a unique evaluation-preserving isomorphism.

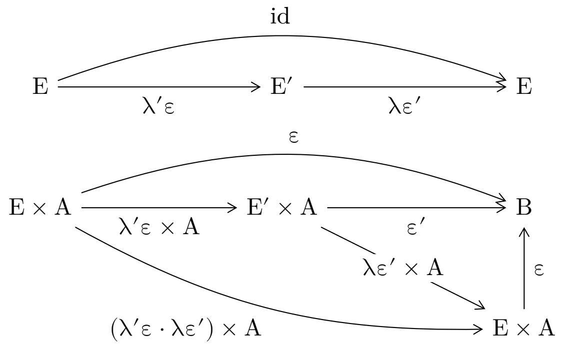
Proof. Suppose that $(E, \varepsilon, \lambda)$ and $(E', \varepsilon', \lambda')$ are both exponentials of A and B. By setting $X := E'$

and $f := \varepsilon'$ in the universal property of E, we have:



That is, $(\lambda\varepsilon' \times \mathrm{A}) \cdot \varepsilon = \varepsilon'$. Symmetrically, by the universal property of E′, we have that $(\lambda'\varepsilon \times \mathrm{A}) \cdot \varepsilon' = \varepsilon$.

We want to show that $\lambda'\varepsilon \cdot \lambda\varepsilon' : \mathrm{E} \to \mathrm{E}$ is the identity map. We do so by uncurrying it. In the following diagram:



we have:

$$\begin{aligned} & ((\lambda' e \cdot \lambda\varepsilon') \times \mathrm{id}\,\mathrm{A}) \cdot \varepsilon \\ = \; & [\text{product functor}] \\ & (\lambda' e \times \mathrm{id}\,\mathrm{A}) \cdot (\lambda\varepsilon' \times \mathrm{id}\,\mathrm{A}) \cdot \varepsilon \\ = \; & [\text{universal property of E}] \\ & (\lambda'\varepsilon \times \mathrm{id}\,\mathrm{A}) \cdot \varepsilon' \\ = \; & [\text{universal property of E}'] \\ & \varepsilon \end{aligned}$$

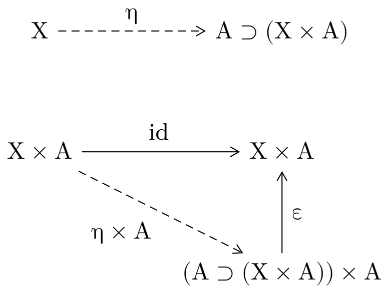
So by identity expansion for exponentials, $\lambda'\varepsilon \cdot \lambda\varepsilon' = \mathrm{id}\,\mathrm{E}$. Similarly, we have $\lambda\varepsilon' \cdot \lambda'\varepsilon = \mathrm{id}\,\mathrm{E}'$.

So $\lambda'\varepsilon : \mathrm{E} \to \mathrm{E}'$ is an isomorphism. By the universal property of E′, it is the only one that respects the evaluation ε'. □

Because exponentials are determined as uniquely as is possible by a behavioral characterization, we write "$\mathrm{A} \supset \mathrm{B}$" to refer to an arbitrary exponential of A and B. The notation "B^{A}" is also common.

Definition 4.4 (pairing map)
For every object A, the universal property of the exponential gives a canonical *pairing map*, which is a higher-order function that pairs an argument with a given parameter:

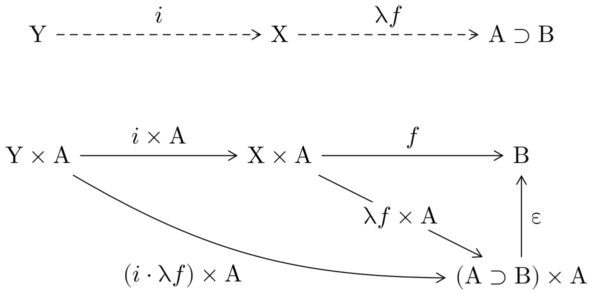
$$\eta(\mathrm{X}) \;:=\; \lambda(\mathrm{id}\,(\mathrm{X} \times \mathrm{A})) \;:\; \mathrm{X} \to \mathrm{A} \supset (\mathrm{X} \times \mathrm{A})$$



Exercise 8 (pre-composing with a curry)
Use the diagram below and the universal property of an exponential object to prove the following:

For an exponential $A \supset B$, an arrow $f : X \times A \to B$ and an arrow $i : Y \to X$,

$$i \cdot \lambda f \quad = \quad \lambda((i \times \mathrm{id}\, A) \cdot f) \quad : \quad Y \to A \supset B$$

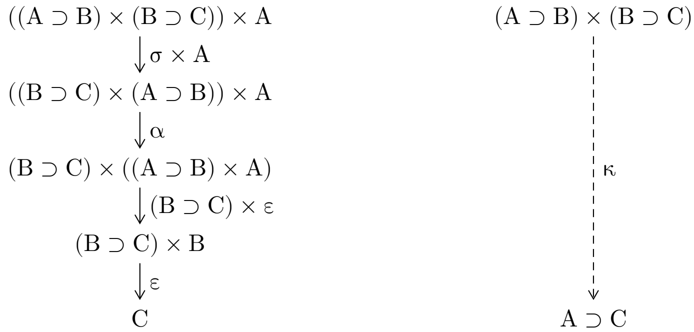


Having products and exponents lets a category talk about its own hom collections, indeed exponential objects are sometimes called "internal homs".

Given any arrow $f : A \to B$, we can precompose it with the isomorphism $1 \times A \to A$ from lemma 2.9 to obtain an arrow $f' : 1 \times A \to B$. We can then curry this arrow to obtain an arrow $\ulcorner f \urcorner := \lambda(f') : 1 \to A \supset B$, yielding a global element of the exponential called the *name* of f.

For any object A, we always have the identity arrow $\mathrm{id}\, A$, and hence a global element $\ulcorner \mathrm{id}\, A \urcorner : 1 \to A \supset A$, the *internal identity*.

Given any objects A, B, C, we can form the composite shown on the left:



where σ is the product symmetry isomorphism (lemma 2.10), α is the product associativity isomorphism (lemma 2.8), and the εs are the respective evaluation maps. Currying this map gives a map $\kappa : (A \supset B) \times (B \supset C) \to A \supset C$ shown on the right, the *internal composition*.

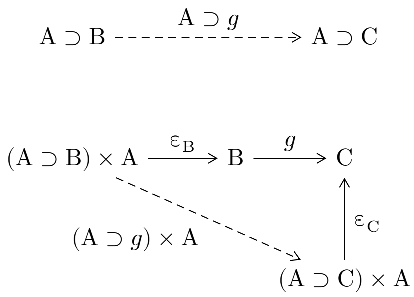
4.2 Exponential Functors

We can use the universal property of exponential objects to define a covariant exponential functor.

Definition 4.5 (covariant exponential of an arrow)
For a fixed object A, we define the *exponential of an arrow* $g : \mathrm{B} \to \mathrm{C}$ to be:

$$\mathrm{A} \supset g \quad := \quad \lambda(\varepsilon_{\mathrm{B}} \cdot g) \quad : \quad \mathrm{A} \supset \mathrm{B} \to \mathrm{A} \supset \mathrm{C}$$

(where we subscript the evaluation maps to match their exponentials). By the universal property of exponentials, $\mathrm{A} \supset g$ is the unique arrow making the triangle commute:



If this definition seems rather unmotivated, it may help to keep in mind that the idea behind an exponential of an arrow, $\mathrm{A} \supset g$, is to somehow "post-compose g at the B inside $\mathrm{A} \supset \mathrm{B}$". This will make more sense shortly, when we will be in a position to see that this definition makes the exponential evaluation into a natural transformation (indeed, the counit of an adjunction). For now, we will content ourselves to observe that it makes $\mathrm{A} \supset -$ into a functor.

Lemma 4.6 (functoriality of exponentials)
The given definition of exponential of arrows yields a functor,

$$\begin{array}{ccc} & \underline{\mathrm{A} \supset -} & \\ \mathbb{C} & \longrightarrow & \mathbb{C} \\ \mathrm{B} & \longmapsto & \mathrm{A} \supset \mathrm{B} \\ g & \longmapsto & \mathrm{A} \supset g \end{array}$$

Proof. In order to prove that $\mathrm{A} \supset -$ is a functor, we must show that it preserves the composition structure.

nullary composition: We must show that

$$\mathrm{A} \supset \mathrm{id}\,\mathrm{B} \;=\; \mathrm{id}\,(\mathrm{A} \supset \mathrm{B})$$

$$\begin{array}{cl} & \mathrm{A} \supset \mathrm{id}\,\mathrm{B} \\ = & [\text{definition of } \mathrm{A} \supset - \text{ on arrows}] \\ & \lambda(\varepsilon_{\mathrm{B}} \cdot \mathrm{id}\,\mathrm{B}) \\ = & [\text{composition unit law}] \\ & \lambda(\varepsilon_{\mathrm{B}}) \\ = & [\text{identity expansion for exponentials}] \\ & \mathrm{id}\,(\mathrm{A} \supset \mathrm{B}) \end{array}$$

binary composition: For consecutive arrows $g : \mathrm{B} \to \mathrm{C}$ and $h : \mathrm{C} \to \mathrm{D}$, we must show that

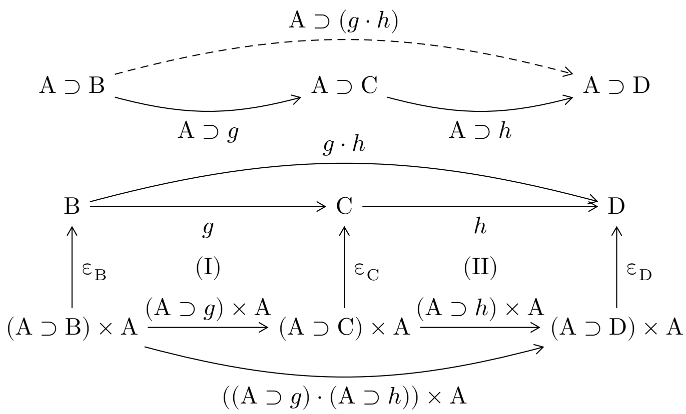
$$\mathrm{A} \supset (g \cdot h) \;=\; (\mathrm{A} \supset g) \cdot (\mathrm{A} \supset h)$$

By definition, $\mathrm{A} \supset (g \cdot h)$ is the unique arrow $f : \mathrm{A} \supset \mathrm{B} \to \mathrm{A} \supset \mathrm{D}$ satisfying $(f \times \mathrm{A}) \cdot \varepsilon_{\mathrm{D}} = \varepsilon_{\mathrm{B}} \cdot g \cdot h$. So we will show that

$$(((\mathrm{A} \supset g) \cdot (\mathrm{A} \supset h)) \times \mathrm{A}) \cdot \varepsilon_{\mathrm{D}} = \varepsilon_{\mathrm{B}} \cdot g \cdot h$$

In the diagram,

$$
\begin{array}{rl}
& (((A \supset g) \cdot (A \supset h)) \times A) \cdot \varepsilon_D \\
= & \text{[functoriality of cartesian product]} \\
& ((A \supset g) \times A) \cdot ((A \supset h) \times A) \cdot \varepsilon_D \\
= & \text{[definition of } A \supset h \text{ – square (II)]} \\
& ((A \supset g) \times A) \cdot \varepsilon_C \cdot h \\
= & \text{[definition of } A \supset g \text{ – square (I)]} \\
& \varepsilon_B \cdot g \cdot h
\end{array}
$$



□

5 Cartesian Closed Categories

A category having all finite products (i.e. a terminal object and binary products), as well as all exponentials, is known as a *cartesian closed category*. A category that additionally has all finite coproducts (i.e. an initial object and binary coproducts) is called *bicartesian closed*.

It turns out that every bicartesian closed category is distributive, with $X \times 0 = 0$ and $X \times (A + B) = (X \times A) + (X \times B)$. We will learn more about such categories later in the course.