Limits and colimits – Part I
Exercise session

Category theory and its applications – ITI9200

https://compose.ioc.ee

10 March 2021

Questions?

(Co)limits in posets

Exercise

Let $(P, \leq)$ be a poset, seen as a thin category.
Answer in **order-theoretic** terms:

1. What is an I-indexed product in P? What is a coproduct?
2. What is a (co)equaliser in P?

What are these when P is the powerset $\mathcal{P}(A)$ of a set A, with $\subseteq$ as order relation?

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- we call a category $\mathcal{C}$ small if it is locally small and $|\mathrm{Ob}(\mathcal{C})| < \kappa$.

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Categories with pullbacks

Exercise

Suppose that $\mathcal{C}$ has pullbacks.

1. Show that, for all $B \in \mathrm{Ob}(\mathcal{C})$, the slice category $\mathcal{C}/_B$ has finite products.
2. Deduce that, if $\mathcal{C}$ has a terminal object, then $\mathcal{C}$ has finite products.
3. Find a category that has pullbacks, but does not have binary products.