Limits and colimits – Part I

Category theory and its applications – ITI9200
https://compose.ioc.ee

8 March 2021

Existence and uniqueness problems

A typical mathematical problem: given an equation

$$f(x) = b$$

or a system of equations

$$\begin{cases} \vdots \\ f_i(x) = b_i \qquad\qquad i \in I \\ \vdots \end{cases}$$

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1. Does there exist a solution?
2. If a solution exists, is it unique?

Existence and uniqueness problems

For a **fixed** $f(x) := (f_i(x))_{i \in I}$, we may ask:

- Does the system

$$\begin{cases} \qquad \vdots \\ f_i(x) = b_i \qquad\qquad i \in I \\ \qquad \vdots \end{cases}$$

  admit a unique solution for **every** choice of $b = (b_i)_{i \in I}$ in a given class $\mathscr{B}$?

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  admit a unique solution for **every** choice of $b = (b_i)_{i \in I}$ in a given class $\mathscr{B}$?

If so, we have a function $b \mapsto \mathrm{u}_f(b)$ giving the unique solution.

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- $b \mapsto \mathrm{u}_f(b)$ is an "encoding" function;
- f is a "decoder": for each $i \in I$ we recover b_i as $f_i(\mathrm{u}_f(b))$.

If f has the existence-and-uniqueness property
with respect to $\mathscr{B}$,
we can encode each family of data $b \in \mathscr{B}$
into the single datum $\mathrm{u}_f(b)$.

Categorical rephrasing

Let's put this kind of problem inside a category $\mathcal{C}$:

$$\begin{cases} \quad \vdots \\ f_i \circ x = b_i \qquad\qquad i \in I \\ \quad \vdots \end{cases}$$

where the f_i, b_i are morphisms in $\mathcal{C}$.

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We can ask if this is true for all meaningful choices of $(b_i)_{i \in I}$...

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to typecheck for all $i \in I$, we need some things to match:

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2. all the b_i must have the same domain $\mathrm{tip}(b)$ (the domain of h);
3. for each $i \in I$, the morphisms f_i and b_i must have the same codomain $F(i)$.

Repackaging these data

- The choice of the objects $F(i)$ for each $i \in I$ is a functor

$$F : I \to \mathcal{C}$$

where the indexing set I is seen as a discrete* category.

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- For each object $C \in \mathrm{Ob}(\mathcal{C})$, let $C!\colon I \to \mathcal{C}$ be the constant functor at C.

Then both f and b define **natural transformations**

$$f\colon \mathrm{tip}(f)! \Rightarrow F, \qquad b\colon \mathrm{tip}(b)! \Rightarrow F$$

whose component at $i \in I$ is f_i and b_i, respectively.

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Cone over $F\colon \mathcal{I} \to \mathcal{C}$

A **cone over** $F\colon \mathcal{I} \to \mathcal{C}$ is the data of

1. an object $\mathrm{tip}(b)$ of $\mathcal{C}$, and
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2. a natural transformation $b\colon \mathrm{tip}(b)! \Rightarrow F$.

In components, a cone b over F is a family of morphisms

$$b_i\colon \mathrm{tip}(b) \to F(i), \qquad i \in \mathrm{Ob}(\mathcal{I})$$

where in addition we have constraints

$$F(h) \circ b_i = b_j$$

for all morphisms $h\colon i \to j$ in $\mathcal{I}$.

Cone over a functor

Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor.

Limit cone

A cone f over F is a **limit cone** if the system of equations

$$f_i \circ x = b_i, \qquad i \in \mathrm{Ob}(\mathcal{I})$$

has a unique solution $\mathrm{u}_f(b)\colon \mathrm{tip}(b) \to \mathrm{tip}(f)$
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We say that F **has a limit** if there exists a limit cone f over F.

If F has a limit f, we can encode *every cone b over F*
(a family of $|\mathrm{Ob}(\mathcal{I})|$-many morphisms)
as a single morphism $\mathrm{u}_f(b) \colon \mathrm{tip}(b) \to \mathrm{tip}(f)$.

Some categorical jargon

- In categorical jargon, existence-and-uniqueness properties are referred to as universal properties.

  You will hear that a limit cone f "has a universal property", and that the solution $\mathrm{u}_f(b)$ of a system of equations is obtained "by universality".

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  This is somewhat improper (the tip is meaningless without the rest of the cone) but it is often done informally.

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- A functor $F\colon \mathcal{I} \to \mathcal{C}$ is often called a diagram of shape $\mathcal{I}$ when we are interested in its limits (or colimits).

  There is no meaningful distinction between a functor and a diagram and this is just one of those traditions that you have to accept.

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Proposition (Essential uniqueness of limits)

Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor, and suppose that both f and g are limit cones over F.
Then $\mathrm{tip}(f)$ and $\mathrm{tip}(g)$ are canonically isomorphic.

Essential uniqueness of limits

Proof

Because f is a limit cone, the system of equations

$$f_i \circ x = g_i, \qquad i \in \mathrm{Ob}(\mathcal{I})$$

has a unique solution $h := \mathrm{u}_f(g) \colon \mathrm{tip}(g) \to \mathrm{tip}(f)$,

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has a unique solution $h := \mathrm{u}_f(g) \colon \mathrm{tip}(g) \to \mathrm{tip}(f)$,
and because g is a limit cone, the system of equations

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has a unique solution $k := \mathrm{u}_g(f) \colon \mathrm{tip}(f) \to \mathrm{tip}(g)$.

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has a unique solution $k := \mathrm{u}_g(f)\colon \mathrm{tip}(f) \to \mathrm{tip}(g)$.
It follows that

$$f_i \circ (h \circ k) = (f_i \circ h) \circ k = g_i \circ k = f_i \qquad \text{for all } i \in \mathrm{Ob}(\mathcal{I})$$
$$g_i \circ (k \circ h) = (g_i \circ k) \circ h = f_i \circ h = g_i \qquad \text{for all } i \in \mathrm{Ob}(\mathcal{I})$$

Essential uniqueness of limits

Proof, cont.d

We have

$$f_i \circ (h \circ k) = f_i \qquad g_i \circ (k \circ h) = g_i$$

for all $i \in \mathrm{Ob}(\mathcal{I})$.

Essential uniqueness of limits

Proof, cont.d

We have

$$f_i \circ (h \circ k) = f_i \qquad g_i \circ (k \circ h) = g_i$$

for all $i \in \mathrm{Ob}(\mathcal{I})$. But also

$$f_i \circ \mathrm{id}_{\mathrm{tip}(f)} = f_i \qquad g_i \circ \mathrm{id}_{\mathrm{tip}(g)} = g_i$$

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for all $i \in \mathrm{Ob}(\mathcal{I})$.

So $h \circ k$ and $\mathrm{id}_{\mathrm{tip}(f)}$ solve the same system of equations, and $k \circ h$ and $\mathrm{id}_{\mathrm{tip}(g)}$ solve the same system of equations. By uniqueness of solutions,

$$h \circ k = \mathrm{id}_{\mathrm{tip}(f)} \qquad k \circ h = \mathrm{id}_{\mathrm{tip}(g)},$$

that is, h and k are two sides of an isomorphism. □

Example: Products

I-indexed product

Let I be a discrete category, i.e. a set, and $F\colon I \to \mathcal{C}$ a functor, i.e. an I-indexed family of objects $(F(i))_{i \in I}$.

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- The component $\pi_j\colon \prod_{i\in I} F(i) \to F(j)$ is called the j^{th} projection.
- When $I = \{1,\dots,n\}$ for some $n > 0$, we may write

$$\prod_{i\in I} F(i) = F(1) \times \ldots \times F(n).$$

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- A cone f over $\imath$ is a limit cone if, for all other cones b, there is a unique morphism $\mathrm{tip}(b) \to \mathrm{tip}(f)$ that satisfies... no equations at all!

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  So cones over $\imath$ correspond bijectively to objects of $\mathcal{C}$.
- A cone f over $\imath$ is a limit cone if, for all other cones b, there is a unique morphism $\mathrm{tip}(b) \to \mathrm{tip}(f)$ that satisfies...
  no equations at all!
- This is the same as an object $\top$ of $\mathcal{C}$ such that, for all other objects c, there is a unique morphism $c \to \top$.
  This is a terminal object of $\mathcal{C}$.

Example: Products in **Set**

Given an I-indexed family of sets $(F(i))_{i\in I}$, the cartesian product

$$\prod_{i\in I} F(i) := \{\langle x_i\rangle_{i\in I} \mid x_i \in F(i)\}$$

together with the projection functions

$$\pi_j \colon \langle x_i\rangle_{i\in I} \mapsto x_j, \qquad j \in I$$

is a product of the $(F(i))_{i\in I}$ in **Set**.

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Proof

Consider a cone b over F, i.e. a family of functions $(b_i \colon \mathrm{tip}(b) \to F(i))_{i \in I}$.

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Consider a cone b over F, i.e. a family of functions $(b_i \colon \mathrm{tip}(b) \to F(i))_{i \in I}$. The system of equations

$$\pi_i \circ x = b_i, \qquad i \in I$$

has the solution

$$\langle b_i \rangle_{i \in I} \colon \mathrm{tip}(b) \to \prod_{i \in I} F(i),$$
$$y \mapsto \langle b_i(y) \rangle_{i \in I}$$

because $\pi_j(\langle b_i(y) \rangle_{i \in I}) = b_j(y)$ for all $y \in \mathrm{tip}(b)$.

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has the solution

$$\begin{aligned} \langle b_i \rangle_{i \in I} \colon \mathrm{tip}(b) &\to \textstyle\prod_{i \in I} F(i), \\ y &\mapsto \langle b_i(y) \rangle_{i \in I} \end{aligned}$$

because $\pi_j(\langle b_i(y) \rangle_{i \in I}) = b_j(y)$ for all $y \in \mathrm{tip}(b)$.

This proves existence of solutions.

Example: Products in **Set**

Proof, cont.d

If $h\colon \mathrm{tip}(b) \to \prod_{i \in I} F(i)$ is another solution,

$$\pi_i(h(y)) = b_i(y)$$

for all $y \in \mathrm{tip}(b)$ and $i \in I$,

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If $h\colon \mathrm{tip}(b) \to \prod_{i\in I} F(i)$ is another solution,

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for all $y \in \mathrm{tip}(b)$ and $i \in I$, so

$$h(y) = \langle \pi_i(h(y))\rangle_{i\in I} = \langle b_i(y)\rangle_{i\in I}$$

for all $y \in \mathrm{tip}(b)$, so as functions $h = \langle b_i\rangle_{i\in I}$.

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for all $y \in \mathrm{tip}(b)$, so as functions $h = \langle b_i \rangle_{i \in I}$.

This proves uniqueness of solutions. □

Categorical structuralism

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- "Any set that behaves like a product of the $(F(i))_{i \in I}$
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$\rightsquigarrow$ essential uniqueness guarantees that any two such sets are canonically isomorphic

Categorical structuralism

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash s : B}{\Gamma \vdash \langle t, s \rangle : A \times B} \qquad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_1 t : A} \qquad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_2 t : B}$$

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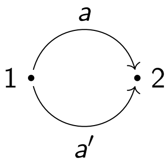
are interpreted as

$$\frac{t \colon \Gamma \to A \qquad s \colon \Gamma \to B}{\langle t, s \rangle \colon \Gamma \to (A \times B)} \qquad \frac{t \colon \Gamma \to (A \times B)}{\pi_1 \circ t \colon \Gamma \to A} \qquad \frac{t \colon \Gamma \to (A \times B)}{\pi_2 \circ t \colon \Gamma \to A}$$

where π is a product of $F(1) = A$ and $F(2) = B$.

Example: Equalisers

Let $\mathcal{P}ar$ be the free category on the graph



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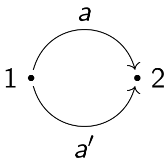

Functors $F\colon \mathcal{P}ar \to \mathcal{C}$ correspond bijectively to
parallel pairs of morphisms

$$F(a)\colon F(1) \to F(2), \qquad F(a')\colon F(1) \to F(2)$$

in $\mathcal{C}$ between the same two objects $F(1), F(2)$.

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Equaliser

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satisfying

$$F(a) \circ b_1 = F(a') \circ b_1 = b_2.$$

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This is the same data as a *single* morphism
$b\colon \mathrm{tip}(b) \to F(1)$ satisfying

$$F(a) \circ b = F(a') \circ b.$$

Example: Equalisers

So an equaliser for $F(a)$ and $F(a')$ is a morphism

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$$e \circ x = b$$

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Example: Equalisers in **Set**

Let $f, g \colon A \to B$ be a parallel pair of functions.

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Let $f, g \colon A \to B$ be a parallel pair of functions.

(This corresponds to the functor $F \colon \mathcal{P}ar \to \mathbf{Set}$ defined by $F(a) := f$, $F(a') := g$.)

The set

$$E_{f,g} := \{x \in A \mid f(x) = g(x)\}$$

together with the subset inclusion $E_{f,g} \hookrightarrow A$ is the equaliser of f, g in **Set**.

Example: An equaliser in **Vect**

A parallel pair is like an equational constraint on elements of A;
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let $A\colon V \to W$ be a linear map of vector spaces.

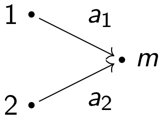
The **nullspace**

$$\ker A = \{v \in V \mid A(v) = 0\}$$

with the inclusion $\ker A \hookrightarrow V$
is the equaliser in **Vect** of A and $0\colon V \to W$.

Example: Pullbacks

Let $\mathcal{C}osp$ be the free category on the graph



Functors $F\colon \mathcal{C}osp \to \mathcal{C}$ correspond bijectively to
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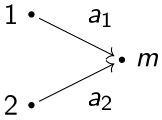

Functors $F\colon \mathcal{C}osp \to \mathcal{C}$ correspond bijectively to
pairs of morphisms with the same codomain
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These are sometimes called **cospans** in $\mathcal{C}$.

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Let $F\colon \mathcal{C}osp \to \mathcal{C}$ be a cospan in $\mathcal{C}$.
A **pullback** of $F(a)$ and $F(a')$ is a limit cone over F.

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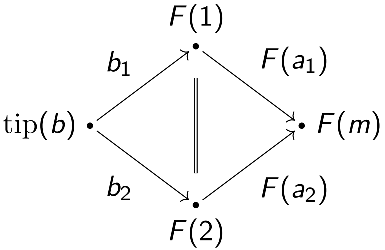
A cone over F is specified by a pair of morphisms

$$b_1\colon \mathrm{tip}(b) \to F(1), \qquad b_2\colon \mathrm{tip}(b) \to F(2)$$

satisfying

$$F(a_1) \circ b_1 = F(a_2) \circ b_2$$

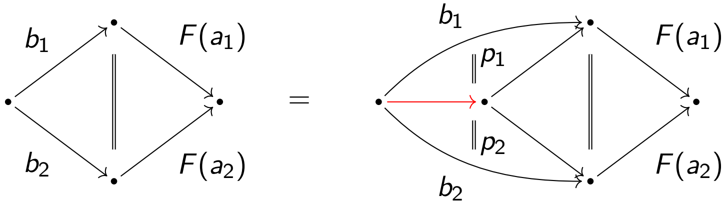


Example: Pullbacks

Among such cones, a pullback has the property that the system of equations

$$p_1 \circ x = b_1, \qquad p_2 \circ x = b_2$$

has a unique solution for all b.



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$$A \times_{f,g} B := \{(x, y) \in A \times B \mid f(x) = g(y)\}$$

together with the functions $p_1 \colon (x, y) \mapsto x$ and $p_2 \colon (x, y) \mapsto y$ is the pullback of f, g in **Set**.

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Notice that this is equal to

- the tip of the equaliser $e\colon E_{f\circ\pi_1, g\circ\pi_2} \hookrightarrow A \times B$,
- together with the functions $\pi_1 \circ e$ and $\pi_2 \circ e$.

Pullbacks from products and equalisers

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Let $f\colon C \to D$, $g\colon C' \to D$ be a cospan in $\mathcal{C}$.

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Let $f\colon C \to D$, $g\colon C' \to D$ be a cospan in $\mathcal{C}$.
Suppose that the following limits exist:

1. a product $C \times C'$ with projections π_1, π_2, and
2. an equaliser $e\colon E_{f\circ\pi_1, g\circ\pi_2} \to C \times C'$ of $f \circ \pi_1$ and $g \circ \pi_2$.

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Then $E_{f\circ\pi_1, g\circ\pi_2}$ together with the morphisms

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is a pullback of f and g.

We can construct limits from other limits!

And now, let's dualise everything

Colimits: a quick definition

The $-^{\mathrm{op}}$ duality turns a functor

$$F : \mathcal{I} \to \mathcal{C}$$

into a functor

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Colimit cone

A cone under F is a colimit cone if it is a limit cone over F^{op}.

Colimits, more slowly

A cone b under $F\colon \mathcal{I} \to \mathcal{C}$ is a family of morphisms

$$b_i\colon F(i) \to \mathrm{tip}(b), \qquad i \in \mathrm{Ob}(\mathcal{I})$$

satisfying the equation

$$b_j \circ F(h) = b_i$$

for all morphisms $h\colon i \to j$ in $\mathcal{I}$.

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Given two cones f, b under F,
we now consider the system of equations

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- We may write $\operatorname{colim} F$ for the (essentially unique!) tip of a colimit cone under F.

Example: Coproducts

I-indexed coproduct

Let I be a discrete category and $F\colon I \to \mathcal{C}$ a functor, i.e. an I-indexed family of objects $(F(i))_{i \in I}$.
A **coproduct** of the family $(F(i))_{i \in I}$ is a colimit cone $\imath$ under F.

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- An $\emptyset$-indexed coproduct is an initial object.

Example: Coproducts in **Set**

Given an I-indexed family of sets $(F(i))_{i \in I}$, the disjoint union

$$\sum_{i \in I} F(i) := \{(i, x) \mid i \in I, x \in F(i)\}$$

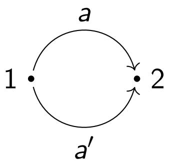
together with the inclusions

$$\imath_j \colon x \mapsto (j, x), \qquad j \in I$$

is a coproduct of the $(F(i))_{i \in I}$ in **Set**.

Example: Coequalisers

Recall that $\mathcal{P}ar$ is



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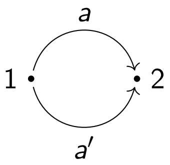

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Let $F\colon \mathcal{P}ar \to \mathcal{C}$ be a parallel pair of morphisms.
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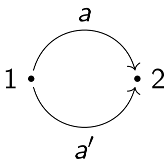

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A **coequaliser** of $F(a)$ and $F(a')$ is a colimit cone q under F.

The data of a cone b under F is fully specified by
a morphism $b\colon F(2) \to \mathrm{tip}(b)$ satisfying

$$b \circ F(a) = b \circ F(a')$$

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So a coequaliser for $F(a)$ and $F(a')$ is a morphism

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The quotient set

$$B/\sim$$

together with the quotient function $q \colon B \twoheadrightarrow B/\sim$
sending $y \in B$ to its equivalence class $[y]_\sim$
is the coequaliser of f, g in **Set**.

Example: Coequalisers in **Set**

Proof

Let $F\colon \mathcal{P}ar \to \mathbf{Set}$ be the functor defined by $F(a) = f$, $F(a') = g$. Consider a cone under F, that is a function $b\colon B \to C$ satisfying

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We let

$$h \colon [y]_\sim \mapsto b(y).$$

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Proof, cont.d

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This proves that if $y \sim y'$, then $b(y) = b(y')$, so h is well-defined.

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Example: Pushouts

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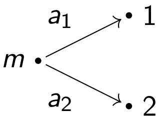

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Dualising the construction of pullbacks from products+equalisers

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Dualising the construction of pullbacks from products+equalisers

The pushout of f and g in **Set** is the quotient

$$A +_{f,g} B := (A + B)/\sim$$

of the coproduct $A + B$ by the equivalence relation $\sim$ generated by

$$\imath_1(f(x)) \approx \imath_2(g(x)) \text{ for all } x \in C,$$

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- When f, g are inclusions of a common subset $C \equiv A \cap B$, then $A +_{f,g} B$ is just the usual union $A \cup B$.