Limits and colimits – Part II
Exercise session

Category theory and its applications – ITI9200
https://compose.ioc.ee

17 March 2021

Questions?

The pullback lemma

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & {\scriptstyle k}\downarrow & & {\scriptstyle \ell}\downarrow \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' \end{array}$$

The pullback lemma

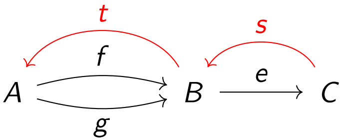
$$\begin{CD}
A @>f>> B @>g>> C \\
@VhVV @VkVV @V\ell VV \\
A' @>>f'> B' @>>g'> C'
\end{CD}$$

Exercise

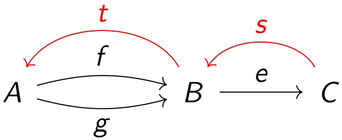
Show that

- if both the left-hand and the right-hand square are pullbacks, then the outer square is a pullback;
- if both the right-hand and the outer square are pullbacks, then the left-hand square is a pullback.

Split coequalisers



Split coequalisers

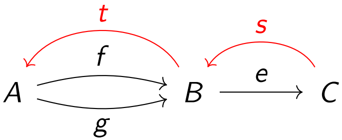


Exercise

Suppose that

1. $e \circ f = e \circ g$ (e is a cone under the pair (f, g))

Split coequalisers

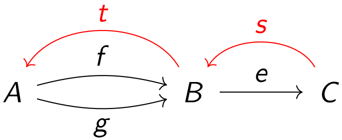


Exercise

Suppose that

1. $e \circ f = e \circ g$ (e is a cone under the pair (f, g))
2. $e \circ s = \mathrm{id}_C$ (s is a section/right inverse of e)

Split coequalisers

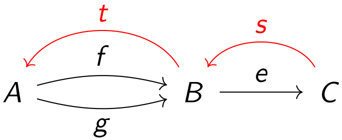


Exercise

Suppose that

1. $e \circ f = e \circ g$ (e is a cone under the pair (f, g))
2. $e \circ s = \mathrm{id}_C$ (s is a section/right inverse of e)
3. $f \circ t = \mathrm{id}_B$ (t is a section/right inverse of f)

Split coequalisers

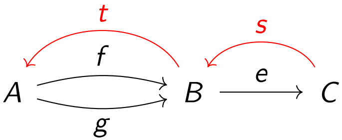


Exercise

Suppose that

1. $e \circ f = e \circ g$ (e is a cone under the pair (f, g))
2. $e \circ s = \mathrm{id}_C$ (s is a section/right inverse of e)
3. $f \circ t = \mathrm{id}_B$ (t is a section/right inverse of f)
4. $g \circ t = s \circ e$.

Split coequalisers



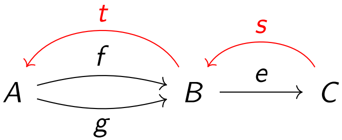
Exercise

Suppose that

1. $e \circ f = e \circ g$ (e is a cone under the pair (f, g))
2. $e \circ s = \mathrm{id}_C$ (s is a section/right inverse of e)
3. $f \circ t = \mathrm{id}_B$ (t is a section/right inverse of f)
4. $g \circ t = s \circ e$.

Prove that e is a coequaliser of (f, g).

Split coequalisers



Exercise

Suppose that

1. $e \circ f = e \circ g$ (e is a cone under the pair (f, g))
2. $e \circ s = \mathrm{id}_C$ (s is a section/right inverse of e)
3. $f \circ t = \mathrm{id}_B$ (t is a section/right inverse of f)
4. $g \circ t = s \circ e$.

Prove that e is a coequaliser of (f, g).
Explain why this colimit is preserved by *every* functor.

Absolute (co)limits

Split coequalisers are examples of **absolute** colimits:

cones that are colimits for "equational" reasons,
hence are preserved by any functor
(because functors preserve equations of morphisms)

Functors that always have a (co)limit

Exercise

Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor.
Suppose that $\mathcal{I}$ has an initial object. Show that F has a limit.