Limits and colimits – Part II

Category theory and its applications – ITI9200
https://compose.ioc.ee

15 March 2021

Categories having (co)limits

Category with $\mathscr{S}$-limits

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ **has $\mathscr{S}$-limits** if, for all categories $\mathcal{I} \in \mathscr{S}$ and functors $F\colon \mathcal{I} \to \mathcal{C}$, there is a limit cone over F in $\mathcal{C}$.

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Limits = products + equalisers

Theorem

Let $\mathcal{C}$ be a category. Suppose that $\mathcal{C}$

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- There is a dual version:
  small coproducts + coequalisers = small colimits

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  cones b over F with $\mathrm{tip}(b) = C$.
- Such a cone is a family of morphisms $b_i : C \to F(i)$
  satisfying the equations

$$F(h) \circ b_i = b_j, \qquad i, j \in \mathrm{Ob}(\mathcal{I}),\ h \in \mathrm{Hom}_{\mathcal{I}}(i, j).$$

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- If we have small products, and $\mathcal{I}$ is small, a family of morphisms with codomain $F(i)$ corresponds to a generalised element of $\prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i)$.

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- If we have small products, and $\mathcal{I}$ is small, a family of morphisms with codomain $F(i)$ corresponds to a generalised element of $\prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i)$.
- If we also have equalisers, we can restrict $\prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i)$ to generalised elements that satisfy equation (1).

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Proof

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Since $\mathcal{C}$ has small products, we can form the products

$$\prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i), \qquad \prod_{\substack{i,j \in \mathrm{Ob}(\mathcal{I}),\\ h \in \mathrm{Hom}_{\mathcal{C}}(i,j)}} F(j)$$

the first indexed by all objects, the second by all *morphisms* of $\mathcal{I}$, with families of projections π_i and π'_h, respectively.

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For each morphism $h\colon i \to j$, we have a pair of morphisms

$$F(h) \circ \pi_i, \quad \pi_j \colon \prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i) \to F(j);$$

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this corresponds to the equation $F(h) \circ b_i = b_j$.

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Proof, cont.d

The families

$$\big(F(h) \circ \pi_i\big)_{\substack{i,j \in \mathrm{Ob}(\mathcal{I}),\\ h \in \mathrm{Hom}_{\mathcal{I}}(i,j)}} \qquad \big(\pi_j\big)_{\substack{i,j \in \mathrm{Ob}(\mathcal{I}),\\ h \in \mathrm{Hom}_{\mathcal{I}}(i,j)}}$$

produce a parallel pair of morphisms

$$\langle F(h) \circ \pi_i \rangle, \quad \langle \pi_j \rangle : \prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i) \to \prod_{\substack{i,j \in \mathrm{Ob}(\mathcal{I}),\\ h \in \mathrm{Hom}_{\mathcal{I}}(i,j)}} F(j);$$

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Let $e \colon \lim F \to \prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i)$ be the equaliser of this parallel pair.

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Proof, cont.d

We claim that $\lim F$ together with $\pi \circ e := (\pi_i \circ e)_{i \in \mathrm{Ob}(\mathcal{I})}$ is a limit cone over F.

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$$F(h) \circ \pi_i \circ e = \pi'_h \circ \langle F(h) \circ \pi_i \rangle \circ e = \pi'_h \circ \langle \pi_j \rangle \circ e = \pi_j \circ e$$

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2. Given another cone b over F, the family $(b_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ induces

$$\langle b_i \rangle \colon \mathrm{tip}(b) \to \prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i)$$

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   We need to show that

$$\langle F(h) \circ \pi_i \rangle \circ \langle b_i \rangle = \langle \pi_j \rangle \circ \langle b_i \rangle$$

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Proof, cont.d

Because b is a cone over F, for all morphisms $h\colon i \to j$, we have

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which implies $\langle F(h) \circ \pi_i \rangle \circ \langle b_i \rangle = \langle \pi_j \rangle \circ \langle b_i \rangle$.

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which implies $\langle F(h) \circ \pi_i \rangle \circ \langle b_i \rangle = \langle \pi_j \rangle \circ \langle b_i \rangle$.

Because e is the equaliser of $\langle F(h) \circ \pi_i \rangle$ and $\langle \pi_j \rangle$,
we obtain a unique morphism

$$k\colon \mathrm{tip}(b) \to \lim F$$

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$$(\pi_i \circ e) \circ k = b_i, \qquad i \in \mathrm{Ob}(\mathcal{I}).$$

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We have constructed $k\colon \mathrm{tip}(b) \to \lim F$ such that

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We have constructed $k \colon \mathrm{tip}(b) \to \lim F$ such that

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This is a solution to the system

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and we can prove that it is unique from the uniqueness properties of products and equalisers.

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This proves that $\pi \circ e$ is a limit cone over F. □

Complete and cocomplete categories

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- When $\mathcal{C}$ is small, the functor category $[\mathcal{C}, \mathbf{Set}]$ has all small limits and colimits.
- A poset P has all limits (resp. colimits) if and only if it has greatest lower bounds (resp. least upper bounds) of arbitrary collections of elements.

Functors and limit cones

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Then Gb with components

$$Gb_i\colon G(\mathrm{tip}(b)) \to GF(i), \qquad i \in \mathrm{Ob}(\mathcal{I}),$$

is a cone over GF.

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The functor G **preserves limits of** F if, whenever f is a limit cone over F, Gf is a limit cone over GF.

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"from limits of GF, we get limits of F"

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Preservation of (co)limits is also very important
in the theory of adjunctions (next lecture!)

Doubly indexed limits

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Suppose that, for each $i \in \mathcal{I}$,
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Then $i \mapsto \lim_{\mathcal{J}} F(i, -)$ extends to a functor

$$\lim_{\mathcal{J}} F\colon \mathcal{I} \to \mathcal{C}.$$

Doubly indexed limits

Sketch of proof

Let $h\colon i \to i'$ be a morphism in $\mathcal{I}$,
let f be the chosen limit cone over $F(i, -)$,
and let f' be the chosen limit cone over $F(i', -)$.

Doubly indexed limits

Sketch of proof

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and let f' be the chosen limit cone over $F(i',-)$.

For each $j \in \mathrm{Ob}(\mathcal{J})$, we have morphisms

$$f_j\colon \lim_{\mathcal{J}} F(i,-) \to F(i,j),$$
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whose composites $(F(h,j) \circ f_j)_{j \in \mathrm{Ob}(\mathcal{J})}$ form
a cone over $F(i',-)$ with tip $\lim_{\mathcal{J}} F(i,-)$.

Doubly indexed limits

Sketch of proof, cont.d

By the universal property of the cone f',
whose tip is $\lim_{\mathcal{J}} F(i', -)$, we obtain a unique morphism

$$\lim_{\mathcal{J}} F(i, -) \to \lim_{\mathcal{J}} F(i', -),$$

and we define $\lim_{\mathcal{J}} F(h, -)$ to be this morphism.

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We can show that this assignment defines a functor (i.e. is compatible with identities and composition). $\square$

Commutativity of (co)limits

If we are in the conditions of the previous proposition, we can ask if $\lim_{\mathcal{J}} F \colon \mathcal{I} \to \mathcal{C}$ has a limit.

Theorem (commutativity of limits)

Suppose that

$$\lim_{\mathcal{J}} F \colon \mathcal{I} \to \mathcal{C}, \qquad \lim_{\mathcal{I}} F \colon \mathcal{J} \to \mathcal{C}$$

are defined and have limit cones with tips

$$\lim_{\mathcal{I}} \lim_{\mathcal{J}} F, \qquad \lim_{\mathcal{J}} \lim_{\mathcal{I}} F.$$

Then $\lim_{\mathcal{I}} \lim_{\mathcal{J}} F$ and $\lim_{\mathcal{J}} \lim_{\mathcal{I}} F$ are canonically isomorphic, and are tips of limit cones over F.

Commutativity of (co)limits

Dually, we have isomorphisms

$$\operatorname{colim}_{\mathcal{I}} \operatorname{colim}_{\mathcal{J}} F \simeq \operatorname{colim}_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F$$

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However limits seldom commute with **colimits**...

Non-commutativity of limits with colimits

Let $F\colon \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor, and suppose

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...but it is not, in general, an isomorphism.

Non-commutativity of limits with colimits

A simple counterexample:
Let $F\colon 2 \times 2 \to \textbf{Set}$ be defined by

$$F(i,j) := 1 \equiv 1_{i,j}$$

for all $i, j \in 2$.

Non-commutativity of limits with colimits

A simple counterexample:
Let $F\colon 2 \times 2 \to \mathbf{Set}$ be defined by

$$F(i,j) \coloneqq 1 \equiv 1_{i,j}$$

for all $i, j \in 2$. Then

$$\operatorname{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F = (1_{1,1} \times 1_{1,2}) + (1_{2,1} \times 1_{2,2}) \simeq 2,$$

$$\lim_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F = (1_{1,1} + 1_{2,1}) \times (1_{1,2} + 1_{2,2}) \simeq 4.$$

Filtered colimits

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A category $\mathcal{C}$ is *filtered* if, for all finite categories $\mathcal{I}$ and functors $F\colon \mathcal{I} \to \mathcal{C}$, there exists a cone under F.

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- A category with finite colimits is filtered.
- A poset is filtered when every finite collection of elements has an upper bound.

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Theorem (Filtered colimits commute with finite limits)

Let $\mathcal{I}$ be a filtered category, $\mathcal{J}$ a finite category, and $F\colon \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ a functor.

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Then the canonical morphism

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is an isomorphism.

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A colimit of such a functor is called a sequential colimit.
For example, an increasing sequence of sets

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$$

defines a functor $\mathbb{N} \to \mathbf{Set}$, whose colimit is

$$\bigcup_{n \in \mathbb{N}} X_n$$

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It follows that sequential colimits commute with finite limits. We recover, for example, the fact that

$$\left(\bigcup_{n\in\mathbb{N}} X_n\right) \times Y \simeq \bigcup_{n\in\mathbb{N}} (X_n \times Y)$$

in **Set**.

Representable property of limits

When $F\colon \mathcal{I} \to \mathcal{C}$ is a functor from a small category, we have

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Theorem

The following are equivalent:

1. F has a limit;
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Interpretation: generalised elements of the limits, i.e. $x \in \mathrm{Hom}_{\mathcal{C}}(C, \lim F)$, are the same as elements $x \in \lim \mathrm{Hom}_{\mathcal{C}}(C, F-)$ of the limit of sets of elements

Representable property of limits

A little teaser of future lectures:
the Yoneda lemma will allow us to identify

$$C \in \mathrm{Ob}(\mathcal{C}) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{C}}(-, C) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

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So $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is a kind of "complete extension" of $\mathcal{C}$.