Category Theory
Lecture 1
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Introduction

Category theory can be thought of as a sort of generalized set theory, where the primitive concepts are those of *set* and *function*, rather than *set* and *membership*. This shift of perspective allows categories to more directly describe many structures, even those that are not very set-like. In category theory, the primitive concept of *set* generalizes to that of *object*, and *function* to *morphism*.

The only assumption that we make about these generalized functions is that they support a *composition structure*, whereby any configuration of compatible morphisms can be combined to yield a new morphism, and the details of how we go about combining the parts into a whole doesn't matter, only the configuration of those parts does.

This is reminiscent of many aspects of our physical world. When we build a castle out of Lego bricks, the order in which we assembled the bricks is not recorded anywhere in the castle, only their configuration with respect to one another remains.

By beginning from very few assumptions, category theory permits a great deal of *axiomatic freedom*. Additional postulates (e.g. the axiom of choice) can then be selectively reintroduced in order to characterize a particular object theory of interest (e.g. set theory).

Because categorical characterizations are based on the concepts of object and morphism, they must describe their subjects *behaviorally* (or externally), rather than *structurally* (or internally): in category theory we can't pin down what the objects of our study actually *are*, only how they relate to one another via the morphisms. In this sense, category theory is the sociology of formal systems.

For example, we will soon see how we can characterize the cartesian product once and for all using a *universal property*. This allows us to describe cartesian products of sets, of groups, of topological spaces, of types, of propositions, and of countless other things, all in one fell swoop, rather than on a tedious case-by-case basis.

1 Basic Categories

1.1 Definition of a Category

Definition 1.1.1 (category)
A **category** $\mathbb{C}$ consists of the following *data*:

- A collection of **objects**, $\mathbb{C}_0$ (comprising the 0-dimensional part of $\mathbb{C}$).
  We write "$\mathrm{A} : \mathbb{C}$" to indicate that $\mathrm{A} \in \mathbb{C}_0$.
- A collection of **morphisms** or "arrows", $\mathbb{C}_1$ (the 1-dimensional part).
  We write "$f :: \mathbb{C}$" to indicate that $f \in \mathbb{C}_1$.
- Two **morphism boundary** functions from arrows to objects:
  domain, "∂^-", and **codomain**, "∂^+".
  For $\mathbb{C}$-objects A and B, we indicate the collection of $\mathbb{C}$-arrows with domain A and codomain B by "$\mathbb{C}\,(\mathrm{A} \to \mathrm{B})$", and call this collection the **hom**[1] from A to B in $\mathbb{C}$. When the category in question is obvious or irrelevant, we just write "$\mathrm{A} \to \mathrm{B}$". We indicate that an arrow f is a member of this collection by writing "$f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$" or "$f : \mathrm{A} \to \mathrm{B}$".
- An **identity morphism** function from objects to arrows, "id", such that both boundaries of an object's identity arrow are just that object itself:

$$\mathrm{id}(\mathrm{A}) : \mathrm{A} \to \mathrm{A}$$

- A partial binary function on arrows, **morphism composition**, "$- \cdot -$", that is defined just in case the codomain of the first is equal to the domain of the second, in which case the composite arrow has the domain of the first and codomain of the second:

$$\text{if} \quad f : \mathrm{A} \to \mathrm{B} \quad \text{and} \quad g : \mathrm{B} \to \mathrm{C} \quad \text{then} \quad f \cdot g : \mathrm{A} \to \mathrm{C}$$

This data is required to respect the following *relations*.

- **composition left unit law**: for an arrow $f : \mathrm{A} \to \mathrm{B}$,

$$\mathrm{id}(\mathrm{A}) \cdot f \quad = \quad f$$

- **composition right unit law**: for an arrow $f : \mathrm{A} \to \mathrm{B}$,

$$f \cdot \mathrm{id}(\mathrm{B}) \quad = \quad f$$

- **composition associative law**: for arrows $f : \mathrm{A} \to \mathrm{B}$, $g : \mathrm{B} \to \mathrm{C}$ and $h : \mathrm{C} \to \mathrm{D}$,

$$(f \cdot g) \cdot h \quad = \quad f \cdot (g \cdot h)$$

By the associative law we may unambiguously write compositions without using brackets, which we will usually do in the sequel.

Definition 1.1.2
In order to avoid gratuitously naming the boundaries of arrows we will give names to some commonly-occurring configurations. We will call a pair of arrows $f, g :: \mathbb{C}$:

[1] presumably, short for "homomorphisms"

- **coinitial**, or a "span", if $\partial^-(f) = \partial^-(g)$,
- **coterminal**, or a "cospan", if $\partial^+(f) = \partial^+(g)$,
- **adjacent** if $\partial^+(f) = \partial^-(g)$,
- **parallel** if both coinitial and coterminal, and
- **anti-parallel** if adjacent in both orderings.

Additionally, we will call an arrow an **endomorphism** if it is adjacent to itself, and a list of arrows a **path** if they are serially adjacent, that is, if $\partial^+(f_i) = \partial^-(f_{i+1})$ for the list $[f_0, \cdots, f_n]$.

Remark 1.1.3 (applicative order composition)
It is common to see the composition $f \cdot g$ written as "$g \circ f$". This can be useful when we want to apply a composite morphism to an argument in a category where a morphism is some sort of function. Then $(g \circ f)(x) = g(f(x))$, which coincides with our custom to write function application with the argument on the right. It may help to read "$f \cdot g$" as "f then g", and to read "$g \circ f$" as "g after f".

Remark 1.1.4 (dimensional promotion)
It is often convenient to call the identity arrow on an object by the same name as the object itself, e.g. to write "A" in place of id(A). This is called **dimensional promotion**, and will become useful as we introduce more complex arrow constructions and concision becomes more of an issue.

Remark 1.1.5 (unbiased presentation)
There is an equivalent presentation of categories in terms of **unbiased composition**. There, instead of a single **binary composition** operation on an adjacent *pair* of arrows, we have a length-indexed composition operation for *paths* of arrows (still with unit and associative laws). In this presentation, an identity morphism is a **nullary composition**, a morphism itself is a **unary composition**, and in general, any length n path of arrows has a unique composite. Although more cumbersome to axiomatize, an unbiased presentation of categories makes it easier to appreciate the idea at the heart of the definition: every composable configuration of things should have a well-defined composite.

Exercise 1.1 (uniqueness of composition units)
By definition, identity arrows act as (two-sided) units for composition. Prove that they are the only arrows with this property.

Hint (proof by *Fight Club*): suppose there were another arrow, $\mathrm{id}'(\mathrm{A}) : \mathrm{A} \to \mathrm{A}$, that acted as a unit for composition at A, then what would we know about the composite $\mathrm{id}(\mathrm{A}) \cdot \mathrm{id}'(\mathrm{A})$?

1.2 Diagrams

We think of an arrow as emanating *from* its domain and proceeding *to* its codomain. We can represent configurations of arrows of a category using a directed graph whose vertices are labeled by objects of the category and whose edges are labeled by arrows.

Such a graph is known as a **diagram**, for example:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

We can represent equations between arrows using diagrams as well. We say that a diagram is **commuting** or "commutes" if the composites of *parallel* paths depicted in the diagram are equal. For example, the fact that each pair of *adjacent* arrows has a unique composite gives us commuting composition triangles, such as:

$$\begin{tikzcd} & B \arrow[dr, bend left, "g"] & \\ A \arrow[ur, bend left, "f"] \arrow[rr, bend right, "f \cdot g"'] & & C \end{tikzcd}$$

Commuting diagrams may be extended by pre- or post-composition of arrows. This is called **whiskering**, and depicts the fact that equality of morphisms is a *congruence* with respect to composition: if $g_1 = g_2$ then $f \cdot g_1 = f \cdot g_2$ and $g_1 \cdot h = g_2 \cdot h$ whenever the composites are defined. The name comes from the fact that the arrows pre- or post-composed to the diagram look like whiskers:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

Pairs of commuting diagrams may also be combined along a common path. This is called **pasting**, and depicts the transitivity of equality: if $f_1 = f_2$ and $f_2 = f_3$ then $f_1 = f_3$.

We may express the *unit* and *associative* laws for composition succinctly using commuting diagrams:

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ A \arrow[ur, "\mathrm{id}"] \arrow[rr, "f" description] \arrow[dr, "f"'] & & B \\ & B \arrow[ur, "\mathrm{id}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[rr, "f \cdot g"] \arrow[dr, "f"'] & & C \arrow[dr, "h"] & \\ & B \arrow[ur, "g"] \arrow[rr, "g \cdot h"'] & & D \end{tikzcd}$$

In the diagram for unitality (left), the triangles representing the left and right unit laws are pasted together along the shared singleton path $[f]$. In the diagram for associativity (right), each of the two composition triangles is whiskered by an arrow (h and f, respectively), and the resulting diagrams are pasted together along the shared path $[f, g, h]$.

In the graphical language of diagrams, any vertex labeled by an object may be duplicated and the two copies joined by an edge labeled by the respective identity morphism. Conversely, any edge labeled by an identity morphism may be collapsed, identifying the two vertices at its boundary, which are necessarily labeled by the same object.

Except for the sake of emphasis, we generally omit composite arrows (including identitites, which are nullary composites) when drawing diagrams, because their existence may always be inferred. Notice that the associative law for composition is built into the graphical language of diagrams by the fact that there is no graphical representation for the bracketing of the arrows in a path.

In order to avoid gratuitously naming objects in diagrams, we will represent an anonymous object as a dot ("$\bullet$"). Two such dots occurring in a diagram need not represent the same object.

Exercise 1.2
Use whiskering and pasting to give a diagrammatic proof that for $f_1, f_2 : \mathrm{A} \to \mathrm{B}$ and $g_1, g_2 : \mathrm{B} \to \mathrm{C}$, if $f_1 = f_2$ and $g_1 = g_2$ then $f_1 \cdot g_1 = f_2 \cdot g_2$.

1.3 Structured Sets as Categories

1.3.1 Discrete Categories

The most trivial possible category has nothing in it. It is called the **empty category**, and written "$\mathbb{0}$". Despite having completely uninteresting *structure*, we will see that this category nevertheless has a very interesting *property*.

Only slightly less trivially, we can consider a category with just a single object, call it "$\star$", and no arrows other than the required identity. This describes a **singleton category**, typically written "$\mathbb{1}$". This category will turn out to have a very interesting property as well.

Generalizing a bit, we can regard any *set* as a category. As a category, a set has its members as objects and no arrows other than the required identities. Categories in which all arrows are identities are called **discrete**.

1.3.2 Preorder Categories

A **preorder** is a reflexive and transitive binary relation on a set, typically written "$- \leq -$". We can interpret a preordered set $(\mathrm{P}\ , \leq)$ as a **preorder category** $\mathbb{P}$ in the following way.

objects: $\mathbb{P}_0 \ \coloneqq\ \mathrm{P}$

arrows: $\mathbb{P}\,(x \to y) \ \coloneqq\ \begin{cases} \{x \leq y\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$

identities: $\mathrm{id}(x) \ \coloneqq\ x \leq x$

composition: $x \leq y \cdot y \leq z \ \coloneqq\ x \leq z$

In other words, a preordered set is a category in which each hom collection is either empty, or else a singleton; and a hom is inhabited just in case its domain is less than or equal to its codomain according to the order relation.

Remark 1.3.1
A preorder need not have anything to do with our usual notion of order on a set. For example, the integers with the "divides" relation, $-|-$, is a perfectly good preordered set, in which $-2 \leq 2$, but also $2 \leq -2$; but not $-2 = 2$ (a preorder need not be *antisymmetric*), nor $2 \leq 3$.

In a preorder category the unit and associative laws of composition are trivially satisfied by the fact that all elements of a singleton or empty set are equal. In fact, every diagram in a preorder category must commute! Preorder categories are sometimes called "thin".

The simplest preorder category that is not discrete has two distinct objects and a single non-identity arrow from one to the other. It looks like this:

$$0 \xrightarrow{\;i\;} 1$$

This category is called the **interval category**, and written "$\mathbb{I}$". It plays an important role in the study of higher-dimensional categorical structures.

1.3.3 Monoid Categories

A **monoid** is a set M together with an associative binary operation "$- * -$" with neutral element "η". We can interpret a monoid $(\mathrm{M}\,, *\,, \eta)$ as a **monoid category** $\mathbb{M}$ in the following way.

objects: $\mathbb{M}_0 \;\coloneqq\; \{\star\}$

arrows: $\mathbb{M}\,(\star \to \star) \;\coloneqq\; \mathrm{M}$

identities: $\mathrm{id}(\star) \;\coloneqq\; \eta$

composition: $x \cdot y \;\coloneqq\; x * y$

Thus, a monoid becomes a category by "suspending" its elements into the hom of *endomorphisms* of an anonymous object, which I imagine looks something like this:

$$\begin{tikzcd} \star \arrow["y", loop above] \arrow["x", loop left] \arrow["z", loop right] \arrow["\cdots", loop below] \end{tikzcd}$$

The unit and associative laws of composition are satisfied by the corresponding laws for the monoid operation.

If we wanted to make the simplest possible monoid category that is not discrete, we would have to think about what it means to be simple. We can begin by postulating a single non-identity arrow, $s : \star \to \star$. But because s is an endomorphism, we must say what $s \cdot s$, $s \cdot s \cdot s$, and in general, $s^{(n)}$ are. One possibility is to introduce no relations. This gives us the free monoid on one generator, better known as $(\mathbb{N}\,, +\,, 0)$.

1.4 Categories of Structured Sets

In addition to (structured) sets *as* categories, we also have categories *of* (structured) sets.

1.4.1 The Category of Sets

There is a **category of sets**, called "Set", whose objects are sets and whose arrows are functions between them. Not surprisingly, we take function composition for the composition of arrows and identity functions for the identity arrows. That is, given composable functions f and g,

$$f \cdot g \quad := \quad \lambda x \,.\, g(f(x)) \qquad \text{and} \qquad \mathrm{id} \quad := \quad \lambda x \,.\, x$$

Composition of SET-morphisms is associative and unital precisely because composition of functions is (check this!).

1.4.2 The Category of Preorders

There is a **category of preorders**, called "PREORD", that has *preordered sets* as objects and monotone (i.e. order-preserving) functions as arrows. Arrow composition is again function composition and the identity arrows are the identity functions.

In order to conclude that this is a category we (i.e. you) must check that the composition of monotone functions is again monotone, and that identity functions are monotone. You just checked that function composition is associative and has identity functions as units, so since monotone functions are functions, you need not check associativity and unitality again for the special case.

1.4.3 The Category of Monoids

The **category of monoids**, MON, has *monoids* as objects and monoid homomorphisms as arrows. A monoid homomorphism is a function between the underlying sets of the monoids that respects the operations and units:

$$\begin{array}{c} f : \textsc{Mon}\,((\mathrm{M}\,,*\,,\eta) \to (\mathrm{M}'\,,*'\,,\eta')) \;:=\; f : \textsc{Set}\,(\mathrm{M} \to \mathrm{M}') \\ \text{such that} \quad f(x * y) \;=\; f(x) *' f(y) \quad \text{and} \quad f(\eta) \;=\; \eta' \end{array}$$

Abstract algebra provides a rich source of categories. These categories generally have sets with some form of algebraic structure as objects and structure-preserving functions as arrows. In addition to that of monoids, we have the category of groups (GRP), of rings (RNG), of modules over a ring, and so on.

1.5 New Categories from Old

Now that we have met a few categories, let's look at some ways to create new categories out of them.

1.5.1 Subcategories

Just as we may restrict our attention to a subset of a given set, we may single out a substructure of a category as well. However, since a category has more structure than a set, we must ensure that the substructure in question remains a category.

Definition 1.5.1 (subcategory)
Given a category $\mathbb{C}$, we may take a **subcategory** $\mathbb{D}$ of $\mathbb{C}$, written "$\mathbb{D} \subseteq \mathbb{C}$" by taking for $\mathbb{D}_0$ a subcollection of $\mathbb{C}_0$ and for $\mathbb{D}_1$ a subcollection of $\mathbb{C}_1$, subject to the restrictions:

- if f is in $\mathbb{D}_1$ then $\partial^-(f)$ and $\partial^+(f)$ are in $\mathbb{D}_0$,
- if A is in $\mathbb{D}_0$ then id(A) is in $\mathbb{D}_1$,
- if f and g are in $\mathbb{D}_1$ and are adjacent then $f \cdot g$ is also in $\mathbb{D}_1$.

The composition structure of arrows when interpreted in $\mathbb{D}$ is the same as in $\mathbb{C}$.

The restrictions are necessary to ensure that the subcollections $\mathbb{D}_0$ and $\mathbb{D}_1$ we choose do, in fact, form a category.

1.5.2 Ordered Pair Categories

Recall that given any two sets, we can form their **set of ordered pairs**:

$$\mathrm{A} \times \mathrm{B} \quad := \quad \{(a\,,b) \mid a \in \mathrm{A} \text{ and } b \in \mathrm{B}\}$$

Likewise, given any two categories, we can construct a new category whose constituent parts are just ordered pairs of the respective parts of the given categories.

Definition 1.5.2 (ordered pair category)
For categories $\mathbb{C}$ and $\mathbb{D}$, the **ordered pair category** $\mathbb{C} \times \mathbb{D}$ has the following structure:

objects: $(\mathrm{A}\,,\mathrm{X}) \quad \text{for} \quad \mathrm{A} : \mathbb{C} \quad \text{and} \quad \mathrm{X} : \mathbb{D}$

arrows: $(f\,,p) \quad \text{for} \quad f :: \mathbb{C} \quad \text{and} \quad p :: \mathbb{D}$,
with $\partial^i((f\,,p)) \;:=\; (\partial^i(f)\,,\partial^i(p))$

identities: $\mathrm{id}((\mathrm{A}\,,\mathrm{X})) \;:=\; (\mathrm{id}(\mathrm{A})\,,\mathrm{id}(\mathrm{X}))$

composition: $(f\,,p) \cdot (g\,,q) \;:=\; (f \cdot g\,,p \cdot q)$

Soon we will be in a position to prove that ordered pair categories have the universal property of a categorical *product*.

1.5.3 Opposite Categories

Recall that each arrow in a category has two boundary objects, its *domain* and *codomain*. Systematically swapping these gives rise to an involutive relation on categories.

Definition 1.5.3 (opposite category)
To any category $\mathbb{C}$, there corresponds an **opposite category**, $\mathbb{C}^\circ$ (pronounced "$\mathbb{C}$-op"), having:

objects: ${\mathbb{C}^\circ}_0 \;:=\; \mathbb{C}_0$

arrows: $\mathbb{C}^\circ(\mathrm{A} \to \mathrm{B}) \;:=\; \mathbb{C}(\mathrm{B} \to \mathrm{A})$

identities: $\mathrm{id}(\mathrm{A}) :: \mathbb{C}^\circ \;:=\; \mathrm{id}(\mathrm{A}) :: \mathbb{C}$

composition: $f \cdot g :: \mathbb{C}^\circ \;:=\; g \cdot f :: \mathbb{C}$

Exercise 1.3
Check that an opposite category satisfies the unit and associative laws of composition, and that the opposite of an opposite category is just the original category.

Despite being simple and purely formal, the opposite category construction is very useful. Because it is an *involution* (for any category $\mathbb{C}$, we have that $(\mathbb{C}^\circ)^\circ = \mathbb{C}$), op is called a **duality**.

For any construction that we may perform in a given category, we can view it from the perspective of the opposite category instead. this determines a **dual construction**. In some cases a construction and its dual may arise within the same category and interact in interesting ways (as, for example, with the *distributive law*).

Moreover, for any proposition that we may state about a given category, there is a **dual proposition** about its opposite category that is true just in case the first proposition is true of the original category. This gives us **dual theorems** for free: in category theory theorems are always two for the proof of one!

1.5.4 Arrow Categories

Definition 1.5.4 (arrow category)
Given a category $\mathbb{C}$, we may derive from it another category, "$\mathbb{C}^{\to}$", known as the **arrow category** of $\mathbb{C}$ with the following structure:

objects: ${\mathbb{C}^{\to}}_0 \;\coloneqq\; \mathbb{C}_1$

arrows: $\mathbb{C}^{\to}\,(f \to g) \;\coloneqq\; \{(i\,,j) \mid i : \mathbb{C}\,(\partial^-(f) \to \partial^-(g)) \text{ and } j : \mathbb{C}\,(\partial^+(f) \to \partial^+(g)) \text{ with } i \cdot g = f \cdot j\}$

identities: $\mathrm{id}(f) \;\coloneqq\; (\mathrm{id}(\partial^-(f))\,,\mathrm{id}(\partial^+(f)))$

composition: $(i\,,j) \cdot (k\,,l) \;\coloneqq\; (i \cdot k\,,j \cdot l)$

In a bit more detail, the objects of $\mathbb{C}^{\to}$ are the arrows of $\mathbb{C}$. Given $\mathbb{C}^{\to}$-objects, $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ and $g : \mathbb{C}\,(\mathrm{C} \to \mathrm{D})$, a $\mathbb{C}^{\to}$-arrow from f to g is a pair of $\mathbb{C}$-arrows, $i : \mathbb{C}\,(\mathrm{A} \to \mathrm{C})$ and $j : \mathbb{C}\,(\mathrm{B} \to \mathrm{D})$ that form a commuting square with f and g in $\mathbb{C}$:

$$\begin{tikzcd} \mathrm{A} \arrow[r, "i"] \arrow[d, "f"'] & \mathrm{C} \arrow[d, "g"] \\ \mathrm{B} \arrow[r, "j"'] & \mathrm{D} \end{tikzcd} \tag{1}$$

Identity $\mathbb{C}^{\to}$-arrows are the commuting $\mathbb{C}$-squares with two opposite sides the same arrow and the other two opposite sides identity arrows. Composition in $\mathbb{C}^{\to}$ is the *pasting* of commuting squares in $\mathbb{C}$:

$$\begin{tikzcd} \mathrm{A} \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & \mathrm{A} \arrow[d, "f"] \\ \mathrm{B} \arrow[r, "\mathrm{id}"'] & \mathrm{B} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{A} \arrow[r, "i"] \arrow[d, "f"'] & \mathrm{C} \arrow[r, "k"] \arrow[d, "g"] & \mathrm{E} \arrow[d, "h"] \\ \mathrm{B} \arrow[r, "j"'] & \mathrm{D} \arrow[r, "l"'] & \mathrm{F} \end{tikzcd}$$

The unit and associative laws of composition are satisfied in $\mathbb{C}^{\to}$ as a consequence of their holding in $\mathbb{C}$ (you should check this). So the arrows of $\mathbb{C}^{\to}$ are the commuting squares of $\mathbb{C}$ (with each commuting $\mathbb{C}$-square represented twice). This tells us something about the 2-dimensional structure of $\mathbb{C}$, namely, which of its squares commute.

We can iterate this construction to explore yet higher-dimensional structure of $\mathbb{C}$. One dimension up, the category $(\mathbb{C}^{\to})^{\to}$ has as objects $\mathbb{C}^{\to}$-arrows (i.e. $\mathbb{C}$-commuting squares) and as arrows $\mathbb{C}^{\to}$-commuting squares. Let's see what these ought to be. A nice way to think about it is to take diagram (1) and imagine that it's actually a 3-dimensional cube that we happen to be seeing orthographically along one face. If we shift our perspective slightly, we will see the following:

$$
\begin{tikzcd}
\bullet \arrow[rr, "i_0"] \arrow[dr, "a"'] \arrow[dd, "f_0"'] & & \bullet \arrow[dr, "c"] \arrow[dd, "g_0"] & \\
& \bullet \arrow[rr, "i_1"] \arrow[dd, "f_1"] & & \bullet \arrow[dd, "g_1"] \\
\bullet \arrow[rr, "j_0"] \arrow[dr, "b"'] & & \bullet \arrow[dr, "d"] & \\
& \bullet \arrow[rr, "j_1"'] & & \bullet
\end{tikzcd}
\tag{2}
$$

We begin with $(\mathbb{C}^{\rightarrow})^{\rightarrow}$-objects f and g, which are actually the $\mathbb{C}^{\rightarrow}$-arrows from a to b and from c to d, respectively. These, in turn, are the $\mathbb{C}$-commuting squares shown on the left and right of diagram (2). Now $(\mathbb{C}^{\rightarrow})^{\rightarrow}$-arrows between these will be $\mathbb{C}^{\rightarrow}$-arrows between their domains and codomains, i and j, which are the $\mathbb{C}$-commuting squares shown on the top and bottom. But there is also the condition that $i \cdot g = f \cdot j$ in $\mathbb{C}^{\rightarrow}$. Composition in $\mathbb{C}^{\rightarrow}$ is pasting in $\mathbb{C}$, and equality of arrows in $\mathbb{C}^{\rightarrow}$ is just pairwise equality in $\mathbb{C}$. So we need that $i_0 \cdot g_0 = f_0 \cdot j_0$ and $i_1 \cdot g_1 = f_1 \cdot j_1$ in $\mathbb{C}$, making the back and front faces commute. In other words, the top and bottom commuting $\mathbb{C}$-squares form a $(\mathbb{C}^{\rightarrow})^{\rightarrow}$-arrow between the left and right commuting $\mathbb{C}$-squares just in case the front and back $\mathbb{C}$-squares commute as well. Then all the paths shown in diagram (2) commute. So $(\mathbb{C}^{\rightarrow})^{\rightarrow}$-arrows are $\mathbb{C}$-commuting cubes.

Exercise 1.4
Recall that the *interval category* is the preorder category with two distinct objects and a single non-identity arrow between them. Describe explicitly each of the objects and arrows of the arrow category of the interval category ($\mathbb{I}^{\rightarrow}$). Is it a preorder category?

1.5.5 Slice Categories

Definition 1.5.5 (slice category)
Given a category $\mathbb{C}$ and object $\mathrm{A} : \mathbb{C}$, there is a category, "$\mathbb{C}/\mathrm{A}$" called the **slice category** of $\mathbb{C}$ over A, with the following structure:

objects: $(\mathbb{C}/\mathrm{A})_0 \;\coloneqq\; \{x :: \mathbb{C} \mid \partial^{+}(x) = \mathrm{A}\}$

arrows: $\mathbb{C}/\mathrm{A}\,(x \to y) \;\coloneqq\; \{p : \mathbb{C}\,(\partial^{-}(x) \to \partial^{-}(y)) \mid p \cdot y = x\}$

Identities and composition are inherited from $\mathbb{C}$.

I imagine the arrows of $\mathbb{C}/\mathrm{A}$ as the bases of inverted triangles with their vertices anchored at A:

$$
\begin{tikzcd}
\mathrm{X} \arrow[rr, "p"] \arrow[dr, "x"'] & & \mathrm{Y} \arrow[dl, "y"] \\
& \mathrm{A} &
\end{tikzcd}
$$

Exercise 1.5
For an object A of a category $\mathbb{C}$, the **coslice category** of $\mathbb{C}$ under A, written "$\mathrm{A}/\mathbb{C}$", is defined as $(\mathbb{C}^{\circ}/\mathrm{A})^{\circ}$. Give an explicit presentation of this category in terms of objects, arrows, identities, and compositions.