Category Theory
Lecture 2
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2 Behavioral Reasoning

A fundamental question we must address when studying any kind of formal system is when two objects with distinct presentations should be considered to be equivalent. We can ask this question about sets, groups, topological spaces, λ-terms, and even categories.

Certainly, whatever relation we choose should be reflexive, transitive and symmetric, and a congruence for certain operations, but beyond that, general guidelines are hard to come by.

For example, we consider two sets to be equivalent if there is a **bijection** between them, that is, if there is an injective and surjective function from one to the other. Recall that a function $p : \mathrm{X} \to \mathrm{Y}$ is **injective** if it "doesn't collapse any elements of its domain":

$$\forall\, x_0, x_1 \in \mathrm{X} \,.\, p(x_0) = p(x_1) \;\supset\; x_0 = x_1 \tag{1}$$

and is **surjective** if it "doesn't miss any elements of its codomain":

$$\forall\, y \in \mathrm{Y} \,.\, \exists\, x \in \mathrm{X} \,.\, p(x) = y \tag{2}$$

We can't translate such element-wise definitions directly to the language of categories because the objects of a category need not be structured sets, so we must find equivalent *behavioral* characterizations.

2.1 Monic and Epic Morphisms

2.1.1 Monomorphisms

In the case of injections, we can do this by rephrasing the property so that rather than discussing the image under p of two points of X, we instead discuss the composition with p of two parallel functions into X.

Lemma 2.1.1
For a function $p : \textsc{Set}\,(\mathrm{X} \to \mathrm{Y})$ the following are equivalent:

(i) p is *injective*

(ii) $\forall\, \mathrm{W} : \textsc{Set};\, f, g : \mathrm{W} \to \mathrm{X};\, w \in \mathrm{W} \,.\, (p \circ f)(w) = (p \circ g)(w) \;\supset\; f(w) = g(w)$

Proof.

$\Rightarrow$ let $x_0 \coloneqq f(w)$ and $x_1 \coloneqq g(w)$

$\Leftarrow$ let $\mathrm{W} \coloneqq \mathbb{1}$ and $f \coloneqq \lambda x \,.\, x_0$ and $g \coloneqq \lambda x \,.\, x_1$.

$\square$

Universal quantification distributes over implication, i.e. if $\forall\, a : \mathrm{A} \,.\, \varphi\ a \ \supset\ \psi\ a$ then $(\forall\, a : \mathrm{A} \,.\, \varphi\ a) \ \supset\ (\forall\, a : \mathrm{A} \,.\, \psi\ a)$. Thus if p is injective then:

$$\forall\, \mathrm{W} : \textsc{Set}; f, g : \mathrm{W} \to \mathrm{X} \,.\, (\forall\, w \in \mathrm{W} \,.\, (p \circ f)(w) = (p \circ g)(w)) \ \supset\ (\forall\, w \in \mathrm{W} \,.\, f(w) = g(w))$$

It may seem that we've just made things worse by introducing two extraneous functions, but now we can use **function extensionality**, the fact that two functions are equal just in case they agree on all points, to rephrase this again, doing away with the points entirely. So if p is injective then:

$$\forall\, \mathrm{W} : \textsc{Set}; f, g : \mathrm{W} \to \mathrm{X} \,.\, f \cdot p = g \cdot p \ \supset\ f = g$$

This property is in fact equivalent to the definition of injective function and is a behavioral characterization that can be stated for any category, not just Set.

Definition 2.1.2 (monomorphism)
An arrow $m :: \mathbb{C}$ is a **monomorphism** (or "monic") if it is *post-cancelable*; that is, if for any arrows $f, g :: \mathbb{C}$,

$$f \cdot m \quad = \quad g \cdot m \qquad \text{implies} \qquad f \quad = \quad g$$

Notice that we are being a bit economical here: in order for f and g to be composable with m, they must be *coterminal*, and in order for their composites with m to be equal, they must also be *coinitial*. So the implication is applicable only to *parallel* f and g adjacent to m, but all that can be inferred.

In diagrams, monomorphisms are conventionally drawn with a tailed arrow: "$\rightarrowtail$".

Lemma 2.1.3 (monics and composition)

- Identity morphisms are monic.
- Composites of monics are monic.
- If the composite $m \cdot n$ is monic then so is m.

Proof.

$$\Rightarrow \begin{array}{l} f \cdot \mathrm{id} \ = \ g \cdot \mathrm{id} \\ [\text{unit law}] \\ f \ = \ g \end{array} \qquad \begin{array}{ll} & f \cdot m \cdot n \ = \ g \cdot m \cdot n \\ \Rightarrow & [n \text{ is monic}] \\ & f \cdot m \ = \ g \cdot m \\ \Rightarrow & [m \text{ is monic}] \\ & f \ = \ g \end{array} \qquad \begin{array}{ll} & f \cdot m \ = \ g \cdot m \\ \Rightarrow & [\text{whiskering}] \\ & f \cdot m \cdot n \ = \ g \cdot m \cdot n \\ \Rightarrow & [m \cdot n \text{ is monic}] \\ & f \ = \ g \end{array}$$

$\square$

Remark 2.1.4 (subobjects)
If we take the *subcategory* of a *slice category* containing just the monomorphisms then we get a *preorder*: given monics $m, n : \mathbb{C}/\mathrm{A}$, we say that $m \leq n$ just in case there is an $f : \mathbb{C}/\mathrm{A}\,(m \to n)$.

$$\begin{tikzcd}
\mathrm{M} \arrow[rr, tail, dashed, "f"] \arrow[dr, tail, "m"'] & & \mathrm{N} \arrow[dl, tail, "n"] \\
& \mathrm{A} &
\end{tikzcd}$$

Such an f, if it exists, is unique because n is monic, and is itself monic by the preceding lemma.

The monics into an object behave very much like the partial order of subsets of a set, in fact, they are known as (representatives of) **subobjects**. This is the beginning of a branch of categorical logic known as **topos theory**, where subobjects are used to interpret logical predicates. However, using a preorder to interpret the entailment relation on propositions sacrifices *proof relevance*: it lets us say *that* one proposition entails another, but not *why* it does so.

2.1.2 Epimorphisms

Using the *op duality*, we can define the property dual to that of being monic. You should check that this amounts to the following:

Definition 2.1.5 (epimorphism)
An arrow $e :: \mathbb{C}$ is an **epimorphism** (or "epic") if it is *pre-cancelable*; that is, if for any arrows $f, g :: \mathbb{C}$,

$$e \cdot f \quad = \quad e \cdot g \qquad \text{implies} \qquad f \quad = \quad g$$

This corresponds to the fact that a surjective function doesn't miss any points in its codomain, so if $p : \mathrm{X} \to \mathrm{Y}$ is surjective then for any parallel $f, g : \mathrm{Y} \to \mathrm{Z}$,

$$(\forall\, x \in \mathrm{X}\ .\ (f \circ p)(x) = (g \circ p)(x)) \quad \supset \quad (\forall\, y \in \mathrm{Y}\ .\ f(y) = g(y))$$

Eliminating the points gives us the definition of epimorphism.

In diagrams, epimorphisms are conventionally drawn with a double-headed arrow: "$\twoheadrightarrow$".

Exercise 2.1
State and prove the *dual theorems* to those in lemma 2.1.3.

In the category Set a function is injective just in case it is monic, and surjective just in case it is epic. In many categories of "structured sets" (e.g. Mon) the monomorphisms are exactly the injective homomorphisms. For instance, the inclusion $(\mathbb{N}\,,+\,,0) \hookrightarrow (\mathbb{Z}, +, 0)$ in the category Mon is a monomorphism. It turns out to be an epimorphism as well, despite not being surjective on its underlying set. So, unlike the situation in Set, in an arbitrary category the existence of a monic and epic morphism between two objects does not suffice to ensure that they are equivalent. Categories where it does suffice are called **balanced**.

2.2 Split Monic and Epic Morphisms

Definition 2.2.1 (split monomorphism)
An arrow s is a **split monomorphism** (or "split monic") if it is post-(semi-)invertible; that is, if there exists an arrow r such that $s \cdot r = \mathrm{id}$.

The dual notion is that of:

Definition 2.2.2 (split epimorphism)
An arrow r is a **split epimorphism** (or "split epic") if it is pre-(semi-)invertible; that is, if there exists an arrow s such that $s \cdot r = \mathrm{id}$.

It would be perverse to name them this way unless split monics were monic and split epics were epic, which indeed they are.

Lemma 2.2.3
A split monomorphism is a monomorphism (and a split epimorphism is an epimorphism).

Proof. Suppose s is split-monic with $s \cdot r = \mathrm{id}$,

$$\begin{array}{ll} & f \cdot s = g \cdot s \\ \Longrightarrow & [\text{whiskering}] \\ & f \cdot s \cdot r = g \cdot s \cdot r \\ \Longrightarrow & [\text{assumption}] \\ & f \cdot \mathrm{id} = g \cdot \mathrm{id} \\ \Longrightarrow & [\text{unit law}] \\ & f = g \end{array}$$

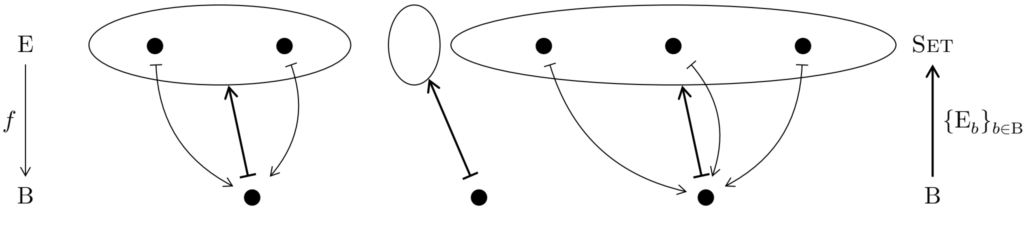
The other case is dual. □

Before moving on, let's consider a particularly pretty application of behavioral reasoning to the axiom of choice. This proposition states that given a family of non-empty sets, there is a function that chooses an element from each one.

We can represent any family of sets with an ordinary function in the following way. Given a function $f : \textsc{Set}\,(\mathrm{E} \to \mathrm{B})$, we can define a function (sometimes called the "fiber"),

$$\begin{array}{rcl} \mathrm{B} & \xrightarrow{f^*} & \wp(\mathrm{E}) \hookrightarrow \textsc{Set} \\ b & \longmapsto & \{e \in \mathrm{E} \mid f(e) = b\} \end{array}$$

And given a family of sets, $\{\mathrm{E}_b\}_{b \in \mathrm{B}}$, which is a disjoint map $\mathrm{B} \to \textsc{Set}$, we can define a *projection function* $\bigsqcup\{\mathrm{E}_b\}_{b\in\mathrm{B}} \to \mathrm{B}$ mapping $e \in \mathrm{E}_b \longmapsto b$. These two constructions are inverse, both the function $f : \mathrm{E} \to \mathrm{B}$ and the family of sets $\{\mathrm{E}_b\}_{b\in\mathrm{B}} : \mathrm{B} \to \textsc{Set}$ just *sort* the elements of E by those in B:



The **axiom of choice** states that if for each $b \in \mathrm{B}$ the set E_b is non-empty then there is a way to choose from E a family of elements $\{e_b\}_{b \in \mathrm{B}}$ such that $\forall\, b \in \mathrm{B}\ .\ f(e_b) = b$ – i.e. such that there is a function $s : \mathrm{B} \to \mathrm{E}$ with $s \cdot f = \mathrm{id}(\mathrm{B})$. Notice that the condition that the sets E_b be non-empty is equivalent to the requirement that f be a surjection. So the axiom of choice asserts that in the category SET, every epimorphism is split!

This is a behavioral characterization of a property that we may ask whether a given category satisfies. For example, because the inclusion $(\mathbb{N}\,,+\,,0) \hookrightarrow (\mathbb{Z}\,,+\,,0)$ is epic in the category MON, it fails to hold there.

Exercise 2.2
The axiom of choice asserts that all epimorphisms in SET are split. Which monomorphisms in SET are split?

Hint: The monomorphisms in SET are exactly the injective functions, and for an injective $p : \mathrm{X} \to \mathrm{Y}$ and $y \in \mathrm{Y}$ there is at most one $x \in \mathrm{X}$ such that $y = p(x)$. For a function $r : \mathrm{Y} \to \mathrm{X}$ to be a post-inverse to p it must send y to x if $y = p(x)$, and this is well-defined by the injectivity of p. But if y is not in the *image* of p then r is free to send it to any element of X.

When we have arrows $s : \mathrm{A} \to \mathrm{B}$ and $r : \mathrm{B} \to \mathrm{A}$ such that $s \cdot r = \mathrm{id}(\mathrm{A})$ we say that s is a **section** of r and that r is a **retraction** of s. So *being* split monic means *having* a retraction, and *being* split epic means *having* a section. We also call $r \cdot s$ a **split idempotent** (it is *idempotent* because $r \cdot s \cdot r \cdot s = r \cdot \mathrm{id}(\mathrm{A}) \cdot s = r \cdot s$).

A section-retraction pair is a *structure* that embodies the *properties* of being monic and epic: the section is split monic, hence monic; while the retraction is split epic, hence epic. Turning properties into structures is one of the recurring themes we encounter in category theory.

Exercise 2.3
Suppose in category $\mathbb{C}$, there is a unique inhabitant f of the hom $\mathrm{A} \to \mathrm{B}$, and that the arrow $a : \mathrm{A}' \to \mathrm{A}$ has retraction a^r and the arrow $b : \mathrm{B} \to \mathrm{B}'$ has section b^s:

$$\begin{tikzcd}
\mathrm{A}' & \mathrm{A} \arrow[l, "a^r"] \arrow[d, "f"] & \mathrm{A}' \arrow[l, "a"] \arrow[ll, bend right, "\mathrm{id}"'] \\
\mathrm{B}' \arrow[r, "b^s"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{B} \arrow[r, "b"] & \mathrm{B}'
\end{tikzcd}$$

Prove that there is a unique inhabitant of the hom $\mathrm{A}' \to \mathrm{B}'$.

Hint: for any arrow $g : \mathrm{A}' \to \mathrm{B}'$, what do we know about $a^r \cdot g \cdot b^s$? What do we find out by whiskering this fact by $a \cdot - \cdot b$?

2.3 Isomorphisms

Lemma 2.3.1
If a morphism has both a section and a retraction then the section and the retraction are equal.

Proof. Given arrow $f : \mathrm{A} \to \mathrm{B}$ with section s and retraction r, by pasting, $s = r$:

$$
\begin{tikzcd}
\mathrm{B} \arrow[r, "s"] \arrow[rrr, bend left=50, "s"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rrr, bend right=50, "r"'] & \mathrm{A} \arrow[r, "f"] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{B} \arrow[r, "r"'] & \mathrm{A}
\end{tikzcd}
$$

□

If $f : \mathrm{A} \to \mathrm{B}$ has section-and-retraction g, then $g : \mathrm{B} \to \mathrm{A}$ necessarily has retraction-and-section f. In other words, f and g are (two-sided) inverses for one another. This leads us to a good behavioral characterization of equivalence in a category.

Definition 2.3.2 (isomorphism)
An arrow $f : \mathrm{A} \to \mathrm{B}$ is an **isomorphism** if there exists an *anti-parallel* arrow $g : \mathrm{B} \to \mathrm{A}$, called an **inverse** of f, that is both retraction and section to f:

$$f \cdot g = \mathrm{id}(\mathrm{A}) \qquad \text{and} \qquad g \cdot f = \mathrm{id}(\mathrm{B})$$

To indicate the existence of an unspecified isomorphism between objects A and B, we write "$\mathrm{A} \cong \mathrm{B}$" and call the objects **isomorphic**. It follows (by *Fight Club*) from lemma 2.3.1 that an inverse of f is unique if it exists, so we can write it unambiguously as "f^{-1}".

Note that although a given arrow can be an isomorphism in at most one way, a given pair of objects can be isomorphic in many different ways. For example, in the category SET every permutation is an **automorphism** (i.e. an isomorphism between an object and itself), and the number of these is the factorial of the number of elements in the set.

Exercise 2.4
Prove the following basic facts about isomorphisms:

- every identity arrow is an isomorphism,
- the composition of two isomorphisms is an isomorphism,
- the inverse of an isomorphism is an isomorphism.

Isomorphism is the right notion of equivalence for objects of an arbitrary category because in categories we must characterize objects behaviorally, and there is generally no way to distinguish objects that behave identically.

Definition 2.3.3 (groupoid)
A category in which every arrow is an isomorphism is called a **groupoid**. In particular, a single-object groupoid is a **group**.

Every category has a maximal subcategory that is a groupoid, defined as follows:

Definition 2.3.4 (groupoid core)
For a category $\mathbb{C}$, its **groupoid core**, $\textsc{Core}\ \mathbb{C}$, is the groupoid with:

objects $(\textsc{Core}\ \mathbb{C})_0 \coloneqq \mathbb{C}_0$

arrows $(\textsc{Core}\ \mathbb{C})_1 \coloneqq \{f :: \mathbb{C} \mid f \text{ an isomrophism}\}$

The composition structure of $\textsc{Core}\ \mathbb{C}$ is inherited from $\mathbb{C}$.

Exercise 2.5
In any category, for adjacent arrows f and g it is a fact that if any two out of $\{f, g, f \cdot g\}$ are isomorphisms, then so is the third. In the previous exercise you proved that the composition of isomorphisms is an isomorphism. In this exercise you will prove the remaining cases.

$$\begin{tikzcd}
 & \mathrm{B} \arrow[dr, "g"] & \\
\mathrm{A} \arrow[ur, "f"] \arrow[rr, "f \cdot g"'] & & \mathrm{C}
\end{tikzcd}$$

(i) If f and $f \cdot g$ are isomorphisms, what is the only arrow g' that we can construct which is *anti-parallel* to g?

(ii) Show that this g' is a section to g (this should be easy).

(iii) Show that g' is a retraction to g. This is a little bit trickier, but note that being a retraction to g is the same thing as being a retraction to $f^{-1} \cdot f \cdot g$ because $f^{-1} \cdot f = \mathrm{id}$.

(iv) It remains to show that if g and $f \cdot g$ are isomorphisms then f is an isomorphism as well. Rather than doing this directly, explain why this is a *dual theorem* to the result that you have just proved.