Category theory course
Lecture 4
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Contents

1 Functors

Category theory has a twofold nature; on one hand, it consists of a systematic study of the rules to generate and taxonomise mathematical objects; on the other hand, it is just another kind of algebraic structure. But then, by this very second reason, and more precisely because for every algebraic structure there is a category of all structures of that sort, there is a category of categories.

Functors are the morphisms of the category of categories, i.e. maps whose defining property is to preserve the category operations: objects go to objects, morphisms to morphisms, and the composition, as well as the identity are preserved.

1.1 Morphisms of categories

Definition 1 (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories; we define a* functor *$F : \mathcal{C} \to \mathcal{D}$ as a pair (F_0, F_1) consisting of the following data:*

- *F_0 is a function $\mathcal{C}_o \to \mathcal{D}_o$ sending an object $C \in \mathcal{C}_o$ to an object $FC \in \mathcal{D}_o$;*
- *F_1 is a family of functions $F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$, one for each pair of objects $A, B \in \mathcal{C}_o$, sending each arrow $f : A \to B$ into an arrow $Ff : FA \to FB$, and such that:*
  - *$F_{AA}(1_A) = 1_{FA}$;*
  - *$F_{AC}(g \circ_{\mathcal{C}} f) = F_{BC}(g) \circ_{\mathcal{D}} F_{AB}(f)$.*

In Haskell, functors are a fundamental construct:

```
-- | All instances of the `Functor` type-class must satisfy two laws.
-- These laws are not checked by the compiler.
--
-- * The law of identity
--   `forall x. (id <$> x) ~ x`
--
-- * The law of composition
--   `forall f g x.(f . g <$> x) ~ (f <$> (g <$> x))`
class Functor k where
-- Pronounced, eff-map.
(<$>) :: (a -> b) -> k a -> k b
```

If $F : \mathcal{C} \to \mathcal{D}$ is a functor in category theory, it corresponds to functor in Haskell, having a data constructor `f :: * -> *` (this is the F_0 part, a *function on objects* from the collection of object $\mathcal{C}_o$ of $\mathcal{C}$ to the collection of objects $\mathcal{D}_o$[1]) and a *function on morphisms*

```
<$> :: (a -> b) -> f a -> f b
```

If you recall how the associativity of `->` works, it is evident that this is a function from the type of maps `a -> b` to the type of maps `f a -> f b`; or rather, this is a function that given a function `u :: a -> b`, and an "element" of type `f a`, yields an element `x :: f b`.

Remark 1. *There is a category whose objects are (small) categories, and whose morphisms $\mathcal{C} \to \mathcal{D}$ are functors. The* identity *functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ of*

[1] Although in Haskell, $\mathcal{C} = \mathcal{D}$ is always the same nameless category `*`.

a category $\mathcal{C}$, and the composition $G \circ F : \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ of two functors are defined in the obvious way; all category axioms follow.[2]

1.2 Examples of functors

As you can imagine, both Haskell and mathematics are crawling with examples of functors;

Example 1 (Examples of functors)**.**

1. *Let's rule out a few edge examples: for every category $\mathcal{C}$, there is a unique functor $F : \varnothing \to \mathcal{C}$, where $\varnothing$ is the empty category with no objects an no morphisms; for every category $\mathcal{C}$, there is an obvious definition of the* identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$*, whose correspondences on objects and on arrows $1_{\mathcal{C},AB} : \mathcal{C}(A,B) \to \mathcal{C}(A,B)$ are all identity functions.*

2. *As we have hinted in the preceding lesson, all correspondences that* forget *a structure, yielding for example the underlying set $U\mathbb{M}$ of a thingum $\mathbb{M}$, are functors $\mathsf{Thng} \to \mathsf{Set}$ from their structured categories of definition to the category of sets and functions.*

3. *All correspondences that* regard *a structure as an example of another are -more or less tautological- examples of functors: a monoid can be seen as a one-object category, and this yields a functor $\boldsymbol{B} : \mathsf{Mon} \to \mathsf{Cat}$ because a functor $F : \boldsymbol{B}M \to \boldsymbol{B}N$ between monoids is but a monoid homomorphism $f : M \to N$; a preset can be seen as a category with at most one arrow between any two objects, and this defines a functor ${}^{\chi}(\_) : \mathsf{Pres} \to \mathsf{Cat}$, because a functor $F : {}^{\chi}P \to {}^{\chi}Q$ between presets is but a monotone function $f : P \to Q$; a set can be seen as a discrete topological space A^{δ} or as a category having only identity arrows, and these are functors $\mathsf{Set} \to \mathsf{Top}, \mathsf{Cat}$. (Continue the list at your will.)*

4. *Let M be a monoid, and $\mathbf{\Delta}$ be the category having objects nonempty, finite, totally ordered sets and monotone functions as morphisms. The correspondence that sends $[n] \in \mathbf{\Delta}$ to the set $M^{[n]}$ of ordered n-tuples $[a_1|\dots|a_n]$ of elements of M and a monotone function $f : [m] \to [n]$ to the function $M^f : M^{[n]} \to M^{[m]}$ defined by*

$$M^f[a_1|\dots|a_n] = [a_{f1}|\dots|a_{fn}]$$

[2] The reader might now wonder if there is a category of Haskell types; unfortunately there is no such thing, but better languages have a better behaviour in this respect (for example `Agda` or `Idris`).

is a functor $\mathbf{\Delta}^{op} \to \mathsf{Set}$*. This functor is called the* classifying space *of the monoid.*

5. *Let* Set_* *be the category of* pointed sets*: objects are pairs* (A, a) *where* $a \in A$ *is a distinguished element, and a morphism* $f : (A, a) \to (B, b)$ *is a function* $f : A \to B$ *such that* $fa = b$*. It is easy to prove that this category is isomorphic to the coslice* $*/\mathsf{Set}$ *of functions* $a : * \to A$*.*

6. *Let* $\partial\mathsf{Set}$ *be the category so defined: objects are sets, and a morphism* $(f, S) : A \to B$ *is a pair* $(S \subseteq A, f : S \to B)$*. This is called a* partial function *from* A *to* B*. It is evident how to define the composition of morphisms, and the identity arrow of* $A \in \partial\mathsf{Set}$*. There is a functor* $(\_)_\bullet : \partial\mathsf{Set} \to \mathsf{Set}_*$ *defined as* $A \mapsto (A \sqcup \{\infty\}, \infty)$*, where* ∞ *is an element that does not belong to* A*, and a function* (f, S) *goes to* $(f, S)_\bullet : A \sqcup \{\infty_A\} \to B \sqcup \{\infty_B\}$*, defined sending* $S^c \cup \{\infty_A\}$ *to* $\{\infty_B\}$*.*

7. *Every directed graph* $\underline{G}$*, with set of vertices* V *and set of edges* E*, defines a quotient set of* V *by the equivalence relation* $\approx$ *generated by the subset of those* (A, B) *for which there is an arrow* $A \to B$*; the symmetric and transitive closure of this relation yields a quotient* V/R *that is usually denoted as the set of* connected components $\pi_0(\underline{G})$*. This is a functor* $\mathsf{Gph} \to \mathsf{Set}$*, and if we now regard a category* $\mathcal{C}$ *as a mere directed graph we obtain a well-defined set* $\pi_0(\mathcal{C})$ *of connected components of a category.*

1.3 Diagrams or: functors as pictures

Most of category theory is based on the intuition that a diagram of a certain shape in a category $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$; the definition of a diagram itself is engineered to blur the distinction between a functor as a morphism of categories, and a functor as a correspondence that "pictures" a small category $\mathcal{J}$ inside a big category $\mathcal{C}$. After having recalled the definition of diagram, and the definition of commutative diagram, we collect a series of examples to convince the reader that this identification of "diagrams as pictures" is fruitful and suggestive.

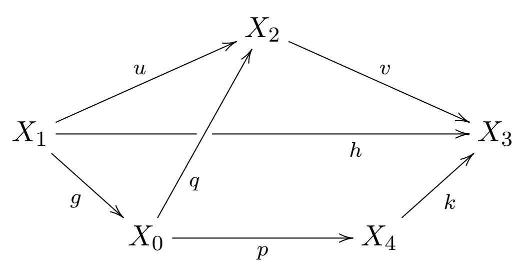
The recommended soundtrack while reading this section is, of course, Emerson, Lake & Palmer's *Pictures at an Exhibition*.

1.3.1 Diagrams, formally

Arrangements of objects and arrows in a category are called *diagrams*; to some extent, category theory is the art of making diagrams *commute*, i.e.

the art of proving that two paths $X \to A_1 \to A_2 \to \cdots \to A_n \to Y$ and $X \to B_1 \to \cdots \to B_m \to Y$ result in the same arrow when they are fully composed.

Remark 2 (Diagrams and their commutation)**.** *Depicting morphisms as arrows allows to draw regions of a given category $\mathcal{C}$ as parts of a (possibly non planar) graph; we call a* diagram *such a region in $\mathcal{C}$, the graph whose vertices are objects of $\mathcal{C}$ and whose edges are morphisms of suitable domains and codomains. For example, we can consider the diagram*



The presence of a composition rule in $\mathcal{C}$ entails that we can meaningfully compose paths $[u_0, \dots, u_n]$ *of morphisms of $\mathcal{C}$. In particular, we can consider diagrams having distinct paths between a fixed source and a fixed 'sink' (say, in the diagram above, we can consider two different paths $\mathfrak{P} = [k, p, g]$ and $\mathfrak{Q} = [v, u]$); both paths go from X_1 to X_3, and we can ask the two compositions $\circ[k, \circ[p, g]] = k \circ (p \circ g)$ and $v \circ u$ to be the same arrow $X_1 \to X_3$; we say that a diagram* commutes at $\mathfrak{P}, \mathfrak{Q}$ *if this is the case; we say that a diagram* commutes *(without mention of $\mathfrak{P}, \mathfrak{Q}$) if it commutes for every choice of paths for which this is meaningful.*

Searching a formalisation of this intuitive pictorial idea leads to the following:

Definition 2. *A* diagram *is a map of directed graphs ('digraphs') $D : J \to \mathcal{C}$ where J is a digraph and $\mathcal{C}$ is the digraph underlying a category.*[3] *Such a diagram D* commutes *if for every pair of parallel edges $f, g : i \rightrightarrows j$ in J one has $Df = Dg$.*

[3] Every small category has an underlying graph, obtained keeping objects and arrows and forgetting all compositions; there is of course a category of graphs, and regarding a category as a graph is another example of forgetful functor. Of course, making this precise means that the collection of categories and functors form a category on its own right.

1.3.2 Diagrams, everywhere

- An object is a diagram: the terminal object in the category of categories is the category
$$[0] = \left\{ \bullet \circlearrowleft 1 \right\}$$
A functor $C : \mathbf{1} \to \mathcal{C}$ consists of an object of $\mathcal{C}$.

- An arrow is a diagram: let
$$[1] = \left\{ 0 \longrightarrow 1 \right\}$$
be the category with two objects $0, 1$ and a single nonidentity morphism $0 \to 1$. A functor $f : \mathbf{2} \to \mathcal{C}$ consists of two objects $X_i = f(i)$ for $i = 0, 1$, and a morphism $X_0 \to X_1$.

- A chain is a diagram: let $n > 2$ be a positive integer. Let
$$[n] = \left\{ 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n \right\}$$
be the category with exactly $n+1$ objects and exactly one nonidentity arrow $j-1 \to j$ for $j = 1, \dots, n$. A functor $c : [n] \to \mathcal{C}$ consists of a tuple of objects $X_0 \to X_1 \to \cdots \to X_n$, and morphisms $X_{j-1} \to X_j$ for $j = 1, \dots, n$.

- A span is a diagram: let
$$\mathcal{S} = \left\{ \begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ 1 & & 2 \end{array} \right\}$$
be the category with three objects and arrows as depicted. A functor $F : \mathcal{S} \to \mathcal{C}$ is a diagram of the same shape, labeled in the same way...

  So, you get the idea. Define a *cospan* in $\mathcal{C}$ to be a functor F with domain the opposite category $\mathcal{S}^{\mathrm{op}}$.

- A commutative square is a diagram: let
$$\begin{array}{ccc} (00) & \longrightarrow & (10) \\ \downarrow & & \downarrow \\ (01) & \longrightarrow & (11) \end{array}$$
be the category with four objects and nonidentity morphisms forming a commutative square. A functor F with this domain, and values in $\mathcal{C}$, consists of a commutative square of objects in $\mathcal{C}$.

- In a similar fashion, a commutative cube is a diagram: a cube is a morphism between the square of sources and the square of targets. Draw a picture, and formalise this statement (what is the identity cube of a square? How do you compose cubes?).
- A diagram is a diagram is a diagram,..., is a diagram: generalise to n-dimensional cubes.

1.4 General definition of limits and colimits

Definition 3 (Cone completions of $\mathcal{J}$)**.** *Let $\mathcal{J}$ be a small category; we denote $\mathcal{J}^{\triangleright}$ the category obtained adding to $\mathcal{J}$ a single terminal object ∞; more in detail, $\mathcal{J}^{\triangleright}$ has objects $\mathcal{J}_o \cup \{\infty\}$, where $\infty \notin \mathcal{J}$, and it is defined by*

$$\mathcal{J}^{\triangleright}(J, J') = \mathcal{J}(J, J')$$
$$\mathcal{J}^{\triangleright}(J, \infty) = \{*\}$$

and it is empty otherwise. This category is called the right cone *of $\mathcal{J}$.*

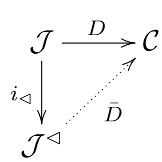
Dually, we define a category $\mathcal{J}^{\triangleleft}$, the left cone *of $\mathcal{J}$, as the category obtained adding to J a single* initial *object $-\infty$; this means that $\mathcal{J}^{\triangleleft}(J, J') = \mathcal{J}(J, J')$, $\mathcal{J}^{\triangleleft}(-\infty, J) = \{*\}$, and it is empty otherwise.*

Example 2. *Let ω be the ordinal number obtained from the union of all finite ordinals; then, when regarded as a category, $\omega^{\triangleleft}$ is the category $\{-\infty \to 0 \to 1 \to \dots\}$ (and thus is isomorphic to ω in an evident way), whereas $\omega^{\triangleright}$ is $\omega + 1$, in the sense of ordinal sum.*

For those willing to embark in an extra exercise: what is $G^{\triangleright}$ if G is a monoid regarded as a category?

Remark 3. *The correspondences $\mathcal{J} \mapsto \mathcal{J}^{\triangleright}$ and $\mathcal{J} \mapsto \mathcal{J}^{\triangleleft}$ are functorial. As an easy exercise, define them on morphisms and prove their functoriality. Prove also that there is a natural embedding $i_{\triangleright}$ of $\mathcal{J}$ into $\mathcal{J}^{\triangleright}$ and one $i_{\triangleleft}$ of $\mathcal{J}$ into $\mathcal{J}^{\triangleleft}$ (that we invariably denote i in the following discussion).*

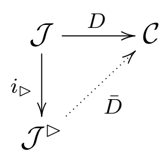
Definition 4 (Cone of a diagram)**.** *Let $\mathcal{J}$ be a small category, $\mathcal{C}$ a category, and $D : \mathcal{J} \to \mathcal{C}$ a functor; all along this section, an idiosyncratic way to refer to D will be as a* diagram *of shape $\mathcal{J}$. We call a* cone *for D any extension of the diagram D to the left cone category of $\mathcal{J}$ defined in , so that the diagram*



commutes.

Every such extension is thus forced to coincide with D on all objects in $\mathcal{J} \subseteq \mathcal{J}^{\lhd}$; the value of $\bar{D}$ on $-\infty$ is called the base of the cone; dually, the value of an extension of D to $\mathcal{J}^{\rhd}$ coincides with D on $\mathcal{J} \subseteq \mathcal{J}^{\rhd}$, and $\bar{D}(\infty)$ is called the *tip* of the cone.

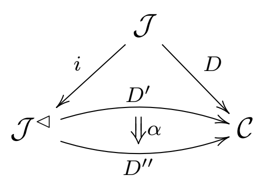
There is of course a similar definition of a *cocone* for D: it is an extension of the diagram D to the right cone category of $\mathcal{J}$ so that the diagram



commutes. Exercise sheds a light on and it substantiates the quote from [**?**] therein: cones for D are exactly cocones for the opposite functor D^{op}.

Remark 4.

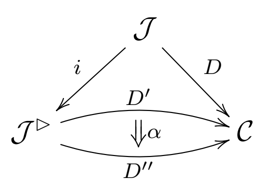
- *The class of cones for D forms a category $Cn(D)$, whose morphisms are the natural transformations $\alpha : D' \Rightarrow D'' : \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i : \mathcal{J} \to \mathcal{J}^{\lhd}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that*



$= 1_D$

as a 2-cell $D \Rightarrow D$.

- *Dually, the class of cocones for D forms a category $Ccn(D)$, whose morphisms are the natural transformations $\alpha : D' \Rightarrow D'' : \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i : \mathcal{J} \to \mathcal{J}^{\rhd}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that*



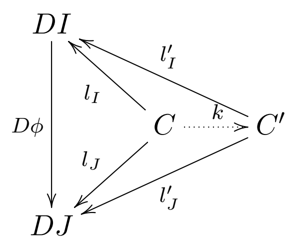
$= 1_D$

as a 2-cell $D \Rightarrow D$.

Definition 5 (Colimit, limit)**.** *The* limit *of a diagram $D : \mathcal{J} \to \mathcal{C}$ is the terminal object denoted '*$\lim_{\mathcal{J}} D$*' in the category of cones for D; dually, the* colimit *of D is the initial object denoted '*$\mathrm{colim}_{\mathcal{J}} D$*' in the category of cocones for D.*

It is a good idea to unwind definitions and in order to obtain the more classically explained notion of limit and colimit: the key for this unwinding operation is that a co/cone for $D : \mathcal{J} \to \mathcal{C}$ amounts to a natural family of maps from a constant object (the base) or to a constant object (the tip).

- A *cone* for a diagram $D : \mathcal{J} \to \mathcal{C}$ is a natural transformation from a constant functor $\Delta_c : \mathcal{J} \to \mathcal{C}$ to $D(\_)$;
- there is a category of cones for $D(\_)$, where morphisms between a cone $c \to D(\_)$ and a cone $C' \to D(\_)$ are arrows $k : C \to C'$ such that the diagram



  is commutative;
- a limit for D is a terminal object in the category of cones for D. This means that given a cone for D, there is a unique arrow k which is a morphism of cones.

Of course, a straightforward dualisation yields the definition of a cocone, and a colimit for D.

1.5 Examples of limits and colimits

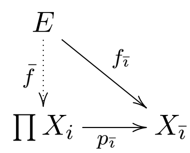
We start with a classical edge example: a terminal object is the limit of the empty diagram.

Example 3. *Let $J = \varnothing$ be the empty set; the limit of the unique diagram $D : \mathcal{J} \to \mathcal{C}$ is the* terminal object *of $\mathcal{C}$.*

The universal property exhibited by the terminal object $*$ of $\mathcal{C}$ is the following: there is a unique morphism $C \to *$ for every $C \in \mathcal{C}$ (with no other condition, since $\mathcal{J}$ is empty).

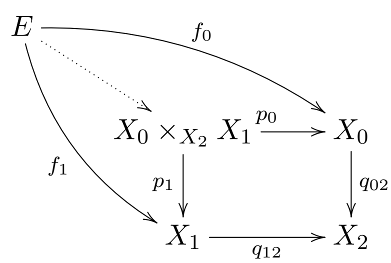
Example 4 (Product)**.** *Let $\mathcal{J}$ be a set, and $\{X_i \mid i \in \mathcal{J}\}$ a family of objects of a category $\mathcal{C}$; the* product *of the $\{X_i\}$'s, denoted $\prod_{i\in\mathcal{J}} X_i$, is the limit of the diagram $D : \mathcal{J} \to \mathcal{C}$, when the set $\mathcal{J}$ is regarded as a discrete category.*

The universal property exhibited by the object $\prod_{i\in\mathcal{J}} X_i$ is the following: there is a cone $\bar{p} = \{p_i : \prod X_j \to X_i \mid i \in \mathcal{J}\}$ such that ($\bar{p}(-\infty) = \prod X_j$ and) for every other cone $\{f_i : E \to X_i \mid i \in \mathcal{J}\}$ there exists a unique dotted $\bar{f} : E \to \prod_{i\in\mathcal{J}} X_i$ such that



commutes for every $\bar{\imath} \in \mathcal{J}$.

Example 5 (Pullback)**.** *Let $\mathcal{J}$ be the category $0 \to 2 \leftarrow 1$, and $\{X_0 \to X_2 \leftarrow X_1\}$ the corresponding diagram $X : \mathcal{J} \to \mathcal{C}$; the* pullback *of the diagram X, denoted $X_0 \times_{X_2} X_1$, is the limit of X; the universal property exhibited by the object $X_0 \times_{X_2} X_1$ is the following: there is a cone $X_0 \xleftarrow{p_0} X_0 \times_{X_2} X_1 \xrightarrow{p_1} X_1$ such that for every other cone $X_0 \xleftarrow{f_0} E \xrightarrow{f_1} X_1$ there exists a unique dotted $\langle f_0, f_1\rangle$ such that*

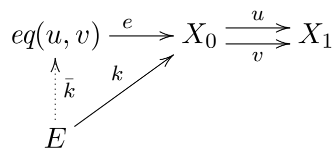


In the same notation above, when we want to stress the dependence of the pullback from the maps q_{02}, q_{12}, the object is sometimes denoted as $q_{02} \times q_{12}$ instead of $X_0 \times_{X_2} X_1$. This is not a real clash of notation, as it is possible to prove that $q_{02} \times q_{12}$ is the product (in the sense of 4 above) of q_{02}, q_{12} regarded as objects of the slice category $\mathcal{C}/X_2$.

In the category of sets, the pullback $X_0 \times_{X_2} X_1$ of a pair f_0, f_1 can be easily characterised as the subset of $X_0 \times X_1$ made by all pairs (x_0, x_1) such that $f_0(x_0) = f_1(x_1)$.

Example 6 (Equaliser)**.** *Let $\mathcal{J}$ be the category $0 \rightrightarrows 1$, and $\{X_0 \overset{u}{\underset{v}{\rightrightarrows}} X_1\}$ the corresponding diagram $X : \mathcal{J} \to \mathcal{C}$; the* equaliser *of the diagram X, denoted*

$eq(u,v)$, is the limit of X; the universal property exhibited by the object is the following: there is an cone $e : eq(u,v) \to X_0$ and for every other cone $k : E \to X_0$ there is a unique dotted $\bar{k} : E \to eq(u,v)$ such that



In the category of sets, the equaliser of a pair of maps u, v can be easily characterised as the subset of X_0 made by all elements such that $u(x) = v(x)$; it is 'the largest subset of X_0 where $u = v$'.

1.5.1 Exercises

Exercises denoted with a star symbol are supposed to be difficult. Don't be put off, and enjoy!

1. Give instances of functors for `(,) a`, `Maybe` and `Either a`. In the first and third case, the functoriality property is "parametric" in the type `a`: this means that for every choice of `a`, there are instances

   ```
   data Pair a b = Pair a b deriving (Eq, Show)

   instance Functor (Pair a) where
     fmap u (Pair a x) = _hole

   data Aut a b = Aut a b deriving (Eq,Show)

   instance Functor (Aut a) where
     fmap u (Aut a x) = _hole
   ```

   Such that

   ```
   (\x -> 7*x) <$> Pair 'a' 3
   => Pair 'a' 21

   (\x -> 7*x) <$> Aut 'a' 3
   => Aut 'a' 21
   ```

   The same proofs show that the corresponding maps $\mathsf{Set} \to \mathsf{Set}$ are indeed functors! Isn't it nice?

2. Show that there is a "tautological" functor $i : \mathsf{Set} \to \mathsf{Cat}$ sending a set to itself, regarded as a discrete category. Given a set A and a category $\mathcal{C}$, show that there is a bijection between the set of functions $\pi_0\mathcal{C} \to A$ (every category can be regarded as a directed graph of some special kind, so it has a set of connected components) and the set of functors $\mathcal{C} \to iA$.

3. ($\star$) Let $F : \mathcal{C} \to \mathcal{Z}$ and $G : \mathcal{D} \to \mathcal{Z}$ be two functors; define the *comma category* of F, G as the category whose

   - objects are arrows in $\mathcal{Z}$ of the form $FC \xrightarrow{f} GD$ (more formally, an object is a tuple $(C, D, f : \mathcal{Z}(FC, GD)))$;
   - morphisms with source $f : FC \to GD$ and target $f' : FC' \to GD'$ are pairs $u : C \to C', v : D \to D'$ such that the square

     $$\begin{array}{ccc} FC & \xrightarrow{f} & GD \\ {\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle Gv} \\ FC' & \xrightarrow[f']{} & GD' \end{array}$$

     is commutative. Show that (F/G) is indeed a category. Show that (F/G) has the following universal property: there is a commutative square

     $$\begin{array}{ccc} (F/G) & \xrightarrow{P} & \mathcal{Z}^{\to} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle J} \\ \mathcal{C} \times \mathcal{D} & \xrightarrow[F \times G]{} & \mathcal{Z} \times \mathcal{Z} \end{array}$$

     and for every other commutative square

     $$\begin{array}{ccc} \mathcal{X} & \xrightarrow{H} & \mathcal{Z}^{\to} \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle J} \\ \mathcal{C} \times \mathcal{D} & \xrightarrow[F \times G]{} & \mathcal{Z} \times \mathcal{Z} \end{array}$$

     there is a unique functor $\langle H, K\rangle : \mathcal{X} \to (F/G)$ such that $P \circ \langle H, K\rangle = H$ and $Q \circ \langle H, K\rangle = K$. In other words, (F/G) is the *pullback* of J and $F \times G$.

4. ($\star$) A morphism $e : X \to X$ in a category $\mathcal{C}$ is said to be *idempotent* if $e \circ e = e$. An object Y is said to be a *retract* of an object X if there exists a commutative diagram

   $$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle r} & \\ X & & = & & X \end{array}$$

   In this case we can identify Y with a subobject of X via the monomorphism i and think of r as a retraction from X onto $Y \subseteq X$. The map ir is an idempotent: $irir = i(ri)r = ir$ by the identity axiom.

   - this idempotent morphism determines Y uniquely (up to isomorphism): show that Y is isomorphic to the equalizer

     $$\mathrm{eq}(ir, 1) \longrightarrow X \underset{1_X}{\overset{ir}{\rightrightarrows}} X$$

- Show that there exists a dual characterization: Y is also the coequalizer of the same pair of maps).

We say that a category $\mathcal{C}$ is *idempotent complete* if every idempotent $e : X \to X$ is of the form ir for some retract $r : X \to Y$ of a $Y \subseteq X$; is the category of finite sets idempotent complete?