the Chu construction & representation of quantum systems

November 6, 2020

Plan

- Understand the papers
  - *Big toy models*: representing physical systems as Chu spaces
  - Coalgebras, Chu spaces, and representations of physical systems

  (mostly the first.)
- Study $\underline{\mathrm{Chu}}(\mathbf{Set}, K)$, in partic. for $K = [0, 1]$ the closed unit interval in $\mathbb{R}$;
- Get a faithful representation of a category of quantum systems;

- Understand projective dualities category-theoretically;
- Replace $[0,1]$ with a smaller (finite?) set while preserving faithfulness of the representation;
- incidentally: define an indexed category $Chu_K^- : \mathbf{Set}^{op} \to \mathbf{CAT}$, whose associated fibration is $\underline{\text{Chu}}(\mathbf{Set}, K)$.

Recall our good old friend $\underline{\text{Chu}}(\textbf{Set}, K) = \underline{\text{Chu}}_K$; objects are

$$\langle A \times X \xrightarrow{e} K \rangle$$

morphisms $e \to e'$ are

$$\begin{array}{ccc} \langle A \times & X \xrightarrow{e} & K \rangle \\ {\scriptstyle u_*}\downarrow & \uparrow{\scriptstyle u^*} & \\ \langle B \times & Y \xrightarrow[e']{} & K \rangle \end{array}$$

An object of $\underline{\text{Chu}}(\textbf{Set}, K)$ is called a *Chu space* over K. We say that a Chu space $\langle A \times X \xrightarrow{e} K \rangle$ is

- extensional if $e(-, x) \equiv e(-, y) \Rightarrow x = y$;
- separated if $e(a, -) \equiv e(b, -) \Rightarrow a = b$;
- biextensional if it is both.

Theorem
There is an adjunction

$$\mathbf{eChu}_K \underset{i}{\overset{R}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{bChu}_K$$

i.e. the subcat of biextensional Chu spaces is reflective in extensional Chu spaces.

The reflection is defined by the biextensional collapse of a Chu Space $\langle X \times A \xrightarrow{e} K \rangle$: we define an equivalence relation on X setting

$$x \simeq_e y \iff e(x, -) \equiv e(y, -)$$

as functions, and set $X^c = X/ \simeq_e$

There is clearly a Chu morphism

$$\begin{array}{ccccc} \langle X & \times & A & \xrightarrow{e} & K\rangle \\ {\scriptstyle u_*}\downarrow & & \uparrow{\scriptstyle u^*} & & \\ \langle X^{\mathrm{c}} & \times & A & \xrightarrow[\bar{e}]{} & K\rangle \end{array}$$

satisfying the universal property of the reflection map: every Chu morphism

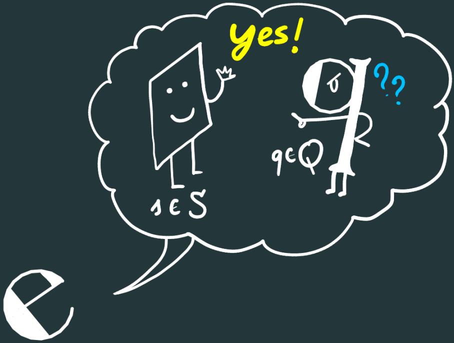
$$\begin{array}{ccccc} \langle X & \times & A & \xrightarrow{e} & K\rangle \\ {\scriptstyle u_*}\downarrow & & \uparrow{\scriptstyle u^*} & & \\ \langle Y & \times & B & \xrightarrow[e']{} & K\rangle \end{array}$$

to a biextensional Chu space $\langle Y \times B \xrightarrow{e'} K\rangle$ descends to a Chu morphism

$$\begin{array}{ccccc} \langle X^{\mathrm{c}} & \times & A & \xrightarrow{\bar{e}} & K\rangle \\ {\scriptstyle u_*}\downarrow & & \uparrow{\scriptstyle u^*} & & \\ \langle Y & \times & B & \xrightarrow[e']{} & K\rangle \end{array}$$

Now let's do Quantum Mechanics: take a set S of states, a set Q of questions you can ask the system S, and a pairing function

$$\langle S \times Q \xrightarrow{e} [0, 1] \rangle$$



Let $\mathcal{H}$ be a complex Hilbert space; define the Chu space

$$\langle \mathcal{H}_o \times L(\mathcal{H}) \xrightarrow{e} [0,1] \rangle$$

where

- $\mathcal{H}_o$ is the preprojective space of $\mathcal{H}$, i.e. the set of nonzero vectors of $\mathcal{H}$;
- $L(\mathcal{H})$ is the lattice of all closed subspaces of $\mathcal{H}$;
- e is the pairing defined by

$$(\psi, S) \longmapsto \frac{(S\psi,\psi)}{(\psi,\psi)} = \frac{(S\psi,\psi)}{\|\psi\|^2} \in [0,1]$$

("normalised expectation value"?)

It is easily seen that

- $e(\psi, S) = e(\frac{\psi}{|\psi|}, S)$, and in particular e is constant over the whole ray of ψ;
- as a consequence, the biextensional collapse $\mathcal{H}_o^c$ of $\langle \mathcal{H}_o^c, L(\mathcal{H}), \bar{e} \rangle$ is the usual projective space $\mathbb{P}\mathcal{H}$; (two vectors are identified if, and only if, they are linearly dependent)

- Given a Chu morphism

$$\begin{array}{ccc} \langle \mathbb{P}\mathcal{H} & \times\ L(\mathcal{H}) & \xrightarrow{\ e\ } [0,1]\rangle \\ u_* \downarrow & \uparrow u^* & \\ \langle \mathbb{P}\mathcal{K} & \times\ L(\mathcal{K}) & \xrightarrow[e']{} [0,1]\rangle \end{array}$$

  one has

$$[\psi] \subseteq u^* S \quad \Longleftrightarrow \quad u_*[\psi] \subseteq S \quad \Longleftrightarrow \quad [u\psi] \subseteq S$$

  where u is the underlying (injective) linear map $\mathcal{H} \to \mathcal{K}$.

- The map u^* in a Chu morphism has an additional left adjoint $u^{\rightarrow}$ defined as the Kan extension

$$u^{\rightarrow}(S) = \bigvee_{\psi \in S_o} u_*[\psi]$$

A representation theorem

Define the following category SymH:

- objects are Hilbert spaces of dimension $d \geq 2$,
- morphisms are emiunitary linear maps (a linear map is emiunitary if it is unitary, $U^\dagger U = 1$, or antiunitary, $U^\dagger U = -1$).

Composition follows the "rule of signs"; every emiunitary map is in particular invertible.

Definition
So defined, $\underline{\mathsf{SymH}}$ is a groupoid; the "groupoid of quantum symmetries".

Theorem
There is a canonical homwise action of S^1 over $\underline{\mathsf{SymH}}$,

$$S^1 \times \underline{\mathsf{SymH}}(\mathcal{H}, \mathcal{K}) : \alpha \mapsto \alpha U$$

that preserves the "sign" of U. The quotient groupoid $_{\mathbb{P}}\underline{\mathsf{SymH}} = \underline{\mathsf{SymH}} /\!\!/ S^1$ is the groupoid of projective quantum symmetries.

There is a functor $R : \underline{\mathsf{SymH}} \to \mathsf{em}\underline{\mathsf{Chu}}_{[0,1]}$ (extensional Chu spaces where a morphism (u_*, u^*) has u_* injective) sending $\left[U \big\downarrow^{\mathcal{H}}_{\mathcal{K}} \right]$ into

$$\begin{array}{ccccc} \langle \mathcal{H}_o & \times & L(\mathcal{K}) & \longrightarrow & K \rangle \\ {\scriptstyle u_*}\big\downarrow & & \big\uparrow{\scriptstyle u^*} & & \\ \langle \mathcal{K}_o & \times & L(\mathcal{K}) & \xrightarrow[e']{} & K \rangle \end{array}$$

where $u_* = U_o$ and $u^* = U^{-1}$. The functor R descends to $\mathbb{P}\underline{\mathsf{SymH}}$ and yields a representation (i.e. a fully faithful functor)

$$\mathbb{P}R : \mathbb{P}\underline{\mathsf{SymH}} \to \mathsf{em}\underline{\mathsf{Chu}}_{[0,1]}.$$

Change of base

Every function $v : K \to L$ induces a functor $v_\bullet : \underline{\mathrm{Chu}}_K \to \underline{\mathrm{Chu}}_L$, sending

$$\langle X \times A \xrightarrow{e} K \rangle \;\mapsto\; \langle X \times A \xrightarrow{ve} L \rangle$$

$v_\bullet$ is always faithful, because it is the identity on morphisms; if v is also injective, then v is also full. Thus, every monomorphism $v : K \to L$ induces a fully faithful functor $\underline{\mathrm{Chu}}_K \to \underline{\mathrm{Chu}}_L$.

This begs the question:

Is there a function $v : [0,1] \to K$, with K finite, such that

$$\mathbb{P}\underline{\mathrm{SymH}} \xrightarrow{QR} bm\underline{Chu}_{[0,1]} \xrightarrow{v_\bullet} bm\underline{Chu}_K$$

Normal Chu spaces as coalgebras

Define an endofunctor of **Set**,

$$F_K(X) := K^{PX}$$

an F_K-coalgebra is a function $\alpha : X \to K^{PX}$, and it defines a normal Chu space $\langle X \times PX \xrightarrow{\in} K \rangle$; an F_K-coalgebra homomorphism is a function $f : X \to Y$ such that

$$\begin{array}{ccc} X & \longrightarrow & K^{PX} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle K^{Pf}} \\ Y & \longrightarrow & K^{PY} \end{array}$$

Theorem
There is an equivalence F_K-CoAlg $\cong$ n<u>Chu</u>$(\textbf{Set}, K)$.

Neat. But...

- Normality is a very restrictive condition on a Chu space;
- F_K is a very ill-behaved funcor, and F_K-coalgebras do not form a nice category (there is no terminal object, no generator);

The reason why this happens is that F_K is not analytic (it does not preserve weak pullbacks).

Bodge
Restrict the cardinality of the powerset functor $P-$ to subsets of cardinality $< \kappa$. Now, $F_{K,\kappa}$ is analytic.

Chu and fibrations

We can do better if we change approach:

- Define a functor $F^Q : X \mapsto \mathbf{Set}(Q, \{\bullet\} \coprod (0,1] \times X)$:

$$q \mapsto \texttt{if } P(q,x) > 0 \quad \texttt{then} \quad (x, P(q,x))$$
$$\texttt{else} \quad \{0\}$$

- think of X as a set of states, and a coalgebra $\alpha : X \to F^Q X$ as sending $(x,q) \in X \times Q$ to $(r,x) \in (0,1] \times X$ if "the probability that the answer to the question q is positive if $r > 0$" and to $\{0\}$ otherwise.

Chu and fibrations

Consider the following two indexed functors:

$$\Psi : \mathbf{Set}^{\mathrm{op}} \to \mathbf{CAT} : Q \mapsto \underline{\mathrm{Chu}}^{(Q)}$$

($\underline{\mathrm{Chu}}^{(Q)}$ is full on Chu morphisms of the form $(u_*, 1_Q)$:

$$\begin{array}{ccc} \langle X \times Q & \xrightarrow{e} & K\rangle \\ u_* \downarrow \quad \uparrow u^* & & \\ \langle Y \times Q & \xrightarrow[e']{} & K\rangle \end{array}$$

it is just the comma category $(Q \times -, \bar{K})$.)

and

$$\Phi : \mathbf{Set}^{\mathrm{op}} \to \mathbf{CAT} : Q \mapsto F^Q\text{-CoAlg}$$

Chu and fibrations

Fact
Every functor $H : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ can be 'straightened' into a pseudofibration

$$\mathcal{C} \int H \longrightarrow \mathcal{C} \ .$$

In $\mathcal{C} \int H$,

- objects are pairs $(C \in \mathcal{C}, A \in HC)$;
- morphisms $(C, A) \to (C', A')$ are pairs $(f : C \to C', \theta : Hf(A') \cong A)$.

Theorem
Straightening $\Psi = \underline{Chu}_K^{(-)}$ *one gets an isomorphism of categories*

$$\mathbf{Set} \int \underline{Chu}_K^{(-)} \cong \underline{Chu}(\mathbf{Set}, K)$$

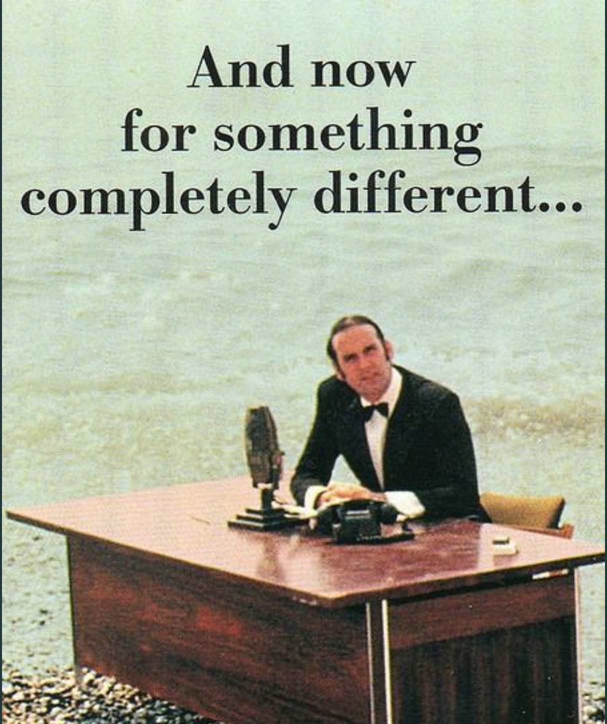
Chu and fibrations

Theorem
There is a fiberwise faithful functor $\tau : \Psi \to \Phi$, whose components are all fully faithful functors

$$\tau_Q : F^Q\text{-}CoAlg \hookrightarrow \underline{Chu}_K^{(Q)}.$$

The components of τ glue to a faithful functor

$$\mathbf{Set} \int \tau : \mathbf{Set} \int \Psi \hookrightarrow \mathbf{Set} \int \Phi$$


And now
for something
completely different...

A Chu-like category of Markov kernels

Recall the definition of a Markov kernel: if $(X, \Sigma_X), (Y, \Sigma_Y)$ are two measurable spaces, it is a map

$$T : \Sigma_X \times Y \to [0, 1]$$

such that

- for each $E \in \Sigma_X$, $T(E, -)$ is a random variable on Y;
- for each $y \in Y$, $T(-, y)$ is a (nice) probability measure on X.

A Chu-like category of Markov kernels

Evidently, each such map is an object of $\underline{\text{Chu}}(\mathit{Set}, [0,1])$.

- Is it possible to use the techniques exposed so far to find an embedding of **Meas** in $\underline{\text{Chu}}(\textbf{Set}, [0,1])$ with nice properties of sorts?
- Is it meaningful the fact that objects of $\underline{\text{Chu}}$ are morphisms of **Meas**, so that morphisms of $\underline{\text{Chu}}(\mathit{Set}, [0,1])$ are (some kind of) 2-cells between measurable spaces?

A Chu-like category of Markov kernels

A measurable function $u : X \to Y$ between measurable spaces goes to the Chu morphisms

$$\begin{array}{ccc} \langle X \times \Sigma_X & \xrightarrow{\delta} & K \rangle \\ u_* \downarrow & \uparrow u^* & \\ \langle Y \times \Sigma_Y & \xrightarrow[\delta]{} & K \rangle \end{array}$$

where $u_* = u$, and $u^* = u^{-1}$ is the inverse image of u. Pairings are "Dirac deltas",

$$\delta(E, x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Space intentionally left blank to chat after the talk