ITI9200

Exercises

V. Brauner, *L'objet qui rêve II*, 1938.

Contents

CHAPTER 0

A word of introduction

As a general rule, solving an exercise in pure Mathematics, the ability to think should be rewarded more than correctness; this means that good ideas leading to wrong answers are more valuable than bad ideas yielding the correct answer.

Category theory follows an even stronger claim: it is based on the belief that the right answer is useless when found through an unenlightening train of thought; you should grow accustomed to this philosophy.

Don't expect all questions to be straightforward. On the contrary, some exercises are meant to be difficult and just an inch above your level; others are meant to force you to learn new things.

I am telling you the Truth and (compatibly with my high tendency to make mistakes) *only* the Truth; but not the whole Truth.

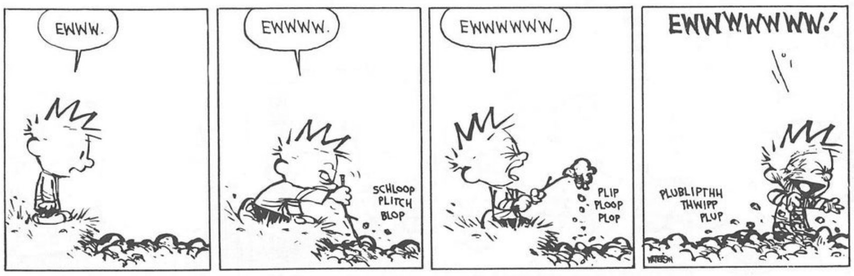

FIGURE 1. You are supposed to get your hands dirty.

Part 1

Mathematics, structurally

Short introduction. In order to proceed more swiftly to the actual core of the course, we assume the reader is familiar with at least the basic notions of set theory: operations on sets (union and intersection of subsets, complement of a subset, cartesian product of two sets,...) and from time to time also with some basic axioms of set theory (for example: 'a set coincides with the collection of its elements', in the sense that $A = B$ has the *meaning* $x \in A \iff a \in B$; and 'for every set X there is a set PX such that $U \in PX$ if and only if $U \subseteq X$'); once this unavoidable core has been taken for granted, however, the aim of the first part of the course is to *reconstruct* the readers' knowledge of elementary mathematics in a tidier, more elegant way –or to be more precise: presenting mathematical ideas paired with a better rationale of *what they are* and *why they are true*.

CHAPTER 1

Orders and relations

Preliminaries. Given sets A, B, we will say that two functions $f, g : A \to B$ **coincide** if they assume the same values elementwise; in symbols,

$$f \equiv g \iff \forall a \in A,\ f(a) = g(a). \tag{1.1}$$

This is usually called the **extensionality principle** for functions.

Let X be a set. One of the axioms of set theory asserts that the following collection

$$PX := \{U \mid U \subseteq X\} \tag{1.2}$$

is a set. It is the set whose elements are exactly all **subsets** of X.[1]

We can build a correspondence between PX and another set: the set whose elements are all functions $f : X \to \{0, 1\}$. In order to define a function

$$c_\bullet :\ PX \longrightarrow 2^X \tag{1.3}$$

just send $U \subseteq X$ to the function $c_U : X \to \{0, 1\}$ sending x to 1 if and only if $x \in U$ (so, since there is no other choice, $c_U(x) = 0$ if and only if $x \notin U$). The function c_U is called the **characteristic function** of the subset $U \subseteq X$.

Proposition 1.1. The correspondence $c_\bullet$ is a function, and it is a bijection, because

- it is injective: when the functions c_U, c_V are the same functions (which means: for every $x \in A$, $c_U(x) = c_V(x)$), then $x \in U$ if and only if $x \in V$, and thus the set U is equal to V;

[1]Recall that a subset U of X is a set with the property that all elements of U are also elements of X: in symbols, $U \subseteq X$ means the formula

$$x \in U \Rightarrow x \in X.$$

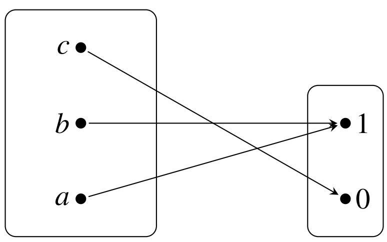

Figure 1. A function $\{a, b, c\} \to \{0, 1\}$: the one sending $a \mapsto 1$, $b \mapsto 1$, $c \mapsto 0$.

- it is surjective, because given any function $f : X \to 2$, f is the characteristic function of the set

$$f^{\leftarrow}(1) := \{x \in X \mid f(x) = 1\}. \tag{1.4}$$

Exercise OR.1$_\square$ Fill in the details in the proof of Proposition 1.1 above. Draw a picture of all functions $\{a, b, c\} \to \{0, 1\}$ and verify that they are $8 = 2^3$, as expected.

Exercise OR.2$_\square$ Solve the same exercise, but backwards: induce from a series of specific examples. If the function $f : \{a, b, c\} \to \{0, 1\}$ of Figure 1 is represented as the triple (110), to denote the fact that the image of a under f is 1, the image of b under f is 1, and the image of c is 0, explain (in words) what's the meaning of the following sentence:

> Here's a list of all functions $\{a, b, c\} \to \{0, 1\}$:
>
> (000) (100) (010) (001) (110) (101) (011) (111).

Surprisingly enough, the above list is an enumeration of all the numbers from 0 to $7 = 2^3 - 1$, written in base 2.

What can you infer about a similar statement regarding the set of functions $\{a, b, c, d\} \to \{0, 1\}$? What can you infer about a similar statement regarding the set of functions $\{a, b, c, d, e\} \to \{0, 1\}$? What can you infer about a similar statement regarding the set of functions $\{a_1, \dots, a_n\} \to \{0, 1\}$?

The weak point of the above 'inductive' approach to discover that the set of all subsets of A is as big as the set of all functions $A \to \{0, 1\}$ is that it cannot be generalised to the case when A is infinite: if A is finite, say with n elements, then PA has 2^n elements; but if A is infinite, what does 2^{infinite} mean –provided it even means anything? **Cardinal arithmetic** is the part of set theory that makes sense of this statement, and others.

Informally speaking, one of the major achievements of Cantor's set theory is to acknowledge the existence of *more than one* infinite set; and in fact, of something more: given *any* infinite, there is a way to build one infinite that is strictly bigger. (Compare this statement with the following fact of life: given a number n, there is a way to build a strictly larger number.)

Definition 1.2. Given a set A, a function $p : A \to \{0, 1\}$ is called a **predicate** or a **proposition**; in mathematical discourse, a predicate is a statement you make about an object A; a proposition is a statement that you make regarding an object, and that you are interested in deeming true or false.

The difference between the two concepts is tenuous, and in fact, you model them mathematically using the same concept: a function $A \to \{0, 1\}$.

Following Proposition 1.1, the predicate/proposition $p : A \to \{0, 1\}$ defines a unique subset of A: the subset of elements of A that make the proposition true.

Following standard practice (it is for example very common in programming) we blur the distinction between the lines of the following table:

Booleans	**answers**	**truth values**
0	no	false
1	yes	true

Table 1. Truth values, booleans, binary answers to questions.

Definition 1.3. A **bijection** between sets X, Y is a function $f : X \to Y$ that is injective and surjective:

- (injective): if for some $x, x' \in X$ we have $f(x) = f(x')$, then $x = x'$;
- (surjective): for every $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$.

Exercise or.3$_\square$ Prove that a function $f : X \to Y$ is bijective if and only if there exists a function $g : Y \to X$ such that for every $y \in Y$, $f(g(y)) = y$ and for every $x \in X$, $g(f(x)) = x$. Such a g is called **inverse** of f; so, f is a bijection if and only if it has an inverse (we also say that f 'is invertible').

Notation 1.4. To denote that there exists *some* unnamed bijection between two sets A, B we often write $A \cong B$. Other synonyms for 'there exists a bijective function $f : A \to B$' are:

- 'the set A can be identified with B';
- 'the set A, or equivalently the set B', and also
- 'the set A, also called the set B'.

(This remark is a half-joke to convey the idea that if $A \cong B$ then the two sets 'behave the same way': each property of A is also enjoyed by B because it can be 'transported' along a bijection $f : A \to B$, and every property of B can be transported along its inverse $f^{-1} : B \to A$.)

Exercise or.4$_\square$ Show that if f has an inverse g as above, g is unique. So, we are allowed to call g *the* inverse of f, when it exists.

Exercise or.5$_\square$ If f is not injective, it cannot be invertible; what is the problem, exactly? What is the obstruction to define the inverse of f? If f is not surjective, it cannot be invertible; again, where is the problem exactly?

Definition 1.5. A set A is called **infinite** (à la Dedekind) if there exist a *proper* subset $U \subset A$ and a bijection $f : U \to A$.[2]

Exercise or.6$_\square$ Welcome to the magic, counterintuitive world of infinite sets! Prove that $\mathbf{N} = \{0, 1, 2, \dots\}$ is infinite à la Dedekind (this was first observed by none less than Galileo). Prove that $\mathbf{Z}$ is infinite à la Dedekind. Write down an explicit bijection between $\mathbf{N}$ and $\mathbf{Z}$; prove that there exists a bijection between $\mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$. Prove that there exists a bijection between the set $\mathbf{N}$ and the set $\{\text{dis}, \text{dat}\} \times \mathbf{N}$, defined as

[2]Note that this definition says when a set is infinite: as counterintuitive as it may seem (because finite sets are 'evidently there' in everyday life, whereas no one can see an infinite one), in Mathematics we explicitly define infinite sets, and we just say that a set is finite if it is not infinite.

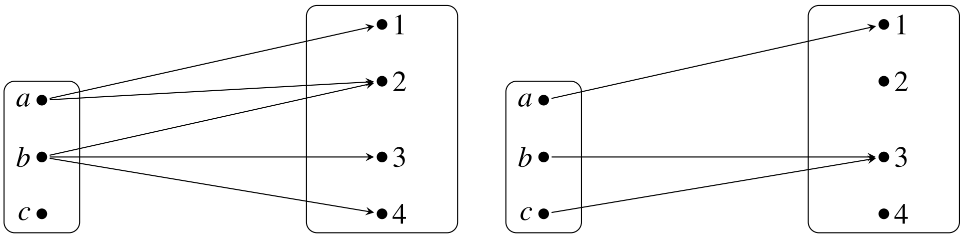

FIGURE 2. On the left, a relation $A = \{a, b, c\} \to \{1, 2, 3, 4\} = B$, as a correspondence between potatoes; on the right, a function $A \to B$.

$$\{\text{DIS}, \text{DAT}\} \times \mathbf{N} := \{(\text{DIS},0), (\text{DIS}, 1), (\text{DIS}, 2), \dots, (\text{DIS}, n), \dots,\\ (\text{DAT}, 0), (\text{DAT}, 1), (\text{DAT}, 2), \dots, (\text{DAT}, n), \dots\} \quad (1.5)$$

EXERCISE OR.7$_\square$ (Very hard, but try to chew this diamond). Let A be an infinite set; prove that there exist a subset E of A and a bijection $f : \mathbf{N} \to E$. So, $\mathbf{N}$ is the **smallest** infinite set, because every other infinite set contains a copy of it.

DEFINITION 1.6. Given sets X, Y, a **relation between** X **and** Y is a subset of the product $X \times Y$; a **relation on** X is a relation between X and itself. The set $\mathsf{Rel}(X, Y)$ of all relations between X and Y then is just the set $P(X \times Y) \cong 2^{X \times Y}$ of all subsets of $X \times Y$.

EXERCISE OR.8$_\square$ Show that there exists a bijection σ between $P(X \times Y)$ and $P(Y \times X)$, induced by a bijection $X \times Y \cong Y \times X$.

Given a relation R between X and Y, the **opposite relation** R^{op} is the image of $R \in P(X \times Y)$ under the bijection σ. The relation R and its opposite carry the exact same amount of information, although formally $R : X \to Y$ and $R^{\text{op}} : Y \to X$. This remark is meant to formalise the idea that the existence of a relation R between sets X, Y is not a 'directed' property (compare this with the fact that a function has instead a specified *domain* and *codomain*).

EXERCISE OR.9$_\square$ Count how many relations there are on a finite set $X = \{x_1, \dots, x_n\}$.

From the very definition of the set $\mathsf{Rel}(X, Y)$ it follows that there exists a natural notion of partial order between relations, defined by $R \le S$ if and only if $R \subseteq S$; thus the intersection (resp., union) of an arbitrary number of relations between X and Y is still a relation. Recall that a **poset**, or a *partial order* or simply an *order* is a set P equipped with a relation $\_ \le \_$ that is reflexive and transitive. A poset $(P, \le)$ is *thin* or *skeletal* if $\_ \le \_$ is also antisymmetric, i.e. if

$$x \le y, y \le x \Rightarrow x = y \quad (1.6)$$

(i.e. there are no nontrivial 'equivalent' element by the equivalence relation $x \equiv y$ defined as $(x \le y) \wedge (y \le x)$).

Examples of thin posets abound (natural numbers with their usual sequential order, natural numbers with the divisibility order, subsets of a given set. . .), as examples of non-thin posets abound. In the following, we will not insist particularly in the difference, and the context (or an easy inspection of the definition) will always allow to understand whether antisymmetry holds. When no further mention is made, a *poset* is meant to be a thin partial order.

Exercise or.10$_\square$ Does the poset $(\mathsf{Rel}(X,Y), \leq)$ has a top element, a bottom element? Define the **complementary** relation of a given $R \in P(X \times Y)$.

Exercise or.11$_\square$ On the relation between relations and functions. Show that a function $f : X \to Y$ is precisely a relation $R \subseteq X \times Y$ with the property that each 'x-section' set $R_x := \{y \in Y \mid (x,y) \in R\}$ is a singleton $\{y\} =: \{y(x)\}$.

The set $Y^X \subseteq \mathsf{Rel}(X,Y)$ is the set of all *functional* relations, i.e. the set of all relations that are functions.

Let $R : X \to Y$ be a functional relation. under which condition the relation R^{op} is a function?

Exercise or.12$_\square$ Let $(P, \leq)$ be a partially ordered set; the **Hasse diagram** of P is the directed graph built in the following way:

- there is a vertex for each element of P;
- there is an edge $q \to p$ connecting p (below) and q (below) if $p \leq q$ and there is no $x \neq p, q$ such that $p \leq x \leq q$.

Draw the Hasse diagram of the following posets:

- $P = \{a, b, c, d\}$ where $a \leq b, a \leq c, b \leq d, c \leq d$;
- $P = 2^A$ where $A = \{0, 1\}$, $P = 2^B$ where $B = \{0, 1, 2\}$, $P = 2^C$ where $C = \{0, 1, 2, 3\}$; do you see a pattern? Generalize.
- P is the set of divisors of 60, ordered by the relation $a \leq b$ if and only if $b = k \cdot a$ for some $k \in \mathbf{N}$.

Definition 1.7. An **algebraic lattice** $(X, \wedge, \vee)$ is a set X equipped with binary operations $\wedge, \vee$ enjoying the following properties: for all $a, b, c \in X$,

- (commutative) $a \wedge b = b \wedge a$ e $a \vee b = b \vee a$;
- (associative) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$;
- (absorption laws) $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Exercise or.13$_\square$ Prove that from the absorption laws it follows that both $\wedge$ and $\vee$ are **idempotent** operations: for all $a \in X$, one has

$$a \wedge a = a \qquad a \vee a = a. \tag{1.7}$$

Exercise or.14$_\square$ If $(X, \wedge, \vee)$ is an algebraic lattice, we can define a partial order relation on X by saying that $a \leq b$ iff $a \wedge b = a$, or equivalently $a \vee b = b$; prove that for every $a, b, c \in X$ the following inequalities hold

D1) $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$;
D2) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Definition 1.8. An algebraic lattice $(X, \wedge, \vee)$ is called **distributive** if the converse inequality in D1 holds.

Figure 3. A delightfully devilish exercise on order theory.

Exercise or.15$_\square$ Prove that the following conditions are equivalent for an algebraic lattice $(X, \wedge, \vee)$:

- X is distributive;
- for every $a, b, c \in X$, $a \vee (b \wedge c) \geq (a \vee b) \wedge (a \vee c)$;
- for every $a, b, c \in X$, $(a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$.

Exercise or.16$_\square$ Let $(X, \wedge, \vee)$ be a distributive lattice with a top and a bottom element; given $a, b, t \in X$, show that there exists at most one $x_{b,t} \in X$ such that $a \wedge x_{b,t} = b$ and $a \vee x_{b,t} = t$. Define the **complement** $\neg a$ of $a \in X$ in a distributive lattice to be $x_{\perp,\top}$. Prove or disprove that $\neg(a \wedge b) = \neg a \vee \neg b$ and $\neg(a \vee b) = \neg a \wedge \neg b$.

Exercise or.17$_\square$ Prove that a lattice $(X, \wedge, \vee)$ is distributive if and only if the following equation is true for every $x, y, z \in X$:

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x). \tag{1.8}$$

Exercise or.18$_\square$ (One implication of this exercise is easy; the other is devilishly difficult. See Figure 3.) Prove that a lattice $(X, \wedge, \vee)$ is distributive if and only if the *cancellation properties* hold: given $y, z \in X$, if there exists $x \in X$ such that $x \wedge z = y \wedge z$ and $x \vee z = y \vee z$, then $y = z$.

Exercise or.19$_\square$ Let $(X, \wedge, \vee)$ be a lattice with a top element $\top$; assume X is totally ordered by the partial order relation associated to the lattice structure, $x \leq y$ if and only if $x \wedge y = x$, if and only if $x \vee y = y$. Define the binary operation $X \times X \to X : (x, y) \mapsto x/y$ by saying that $x/y = \top$ if $x \leq y$ and b otherwise. Prove that for every $x, y, z \in X$ we have

$$x \wedge y \leq z \quad \text{if and only if} \quad x \leq y/z. \tag{1.9}$$

A relation R on a set X is called

- **reflexive** if for every $x \in X$ we have $(x, x) \in R$;
- **symmetric** if for every $x, y \in X$ we have that, if $(x, y) \in R$, then $(y, x) \in R$;
- **transitive** if for every $x, y, z \in X$ we have that, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

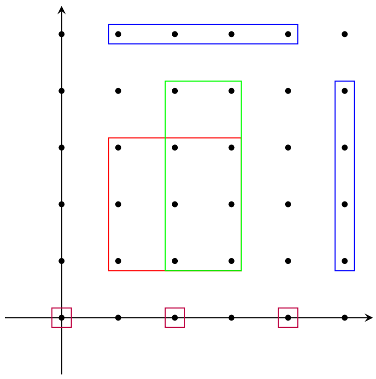

FIGURE 4. The **graph** of a relation depicts the subset $R \subseteq X \times X$ in the 'plane' $X \times X$. Here you see the graph of a few relations on the set $\mathbf{N}$, in different colours.

EXERCISE OR.20 $\square$ Describe the structure of the set $\mathsf{rRel}(X)$ of reflexive relations on X, of the set $\mathsf{sRel}(X)$ of symmetric relations on X, and of the set $\mathsf{tRel}(X)$ of transitive relations: is the intersection (resp., union) of an arbitrary number of elements in $\mathsf{rRel}(X), \mathsf{sRel}(X), \mathsf{tRel}(X)$, still an element of $\mathsf{rRel}(X), \mathsf{sRel}(X), \mathsf{tRel}(X)$?

EXERCISE OR.21 $\square$ On the graphical representation of a relation. Let X be a set; a relation R on X can be depicted as in Figure 4, as a subset of the Cartesian product $X \times X$. This allows for a graphical representation of properties of R. Show that a relation R is reflexive if and only if it contains the diagonal. Show that a relation R is symmetric if and only if it is symmetric with respect to the diagonal. Find a similar graphical interpretation for the transitive property.

EXERCISE OR.22 $\square$ Find a relation R on a set X that is

- reflexive and symmetric, but not transitive;
- symmetric and transitive, but not reflexive;
- reflexive and transitive, but not symmetric;
- reflexive, but neither symmetric nor transitive;
- symmetric, but neither reflexive nor transitive;
- transitive, but neither reflexive nor symmetric;
- not symmetric, not transitive, not reflexive.

(You can choose different sets X for each item of the list.)

EXERCISE OR.23 $\square$ Does the poset of reflexive relations on a set X admit a top element? A bottom element? Same question with the poset of symmetric relations; same question with the poset of transitive relations.

EXERCISE OR.24 $\square$ A generalization of the order on a powerset. Let U be a set called **universe**; consider the set of **multisets** in U, i.e. sequences $\langle x_1, \dots, x_n \rangle$ of **possibly repeated** elements of U, irregardless of order.[3] The set of all multisets in a given U is denoted $\mathcal{M}(U)$.

[3]This means that the multiset $\langle 1, 1, 2 \rangle$ is considered equal to the multiset $\langle 1, 2, 1 \rangle$, but not to the multiset $\langle 1, 2 \rangle$. Compare what happens, instead, with sets.

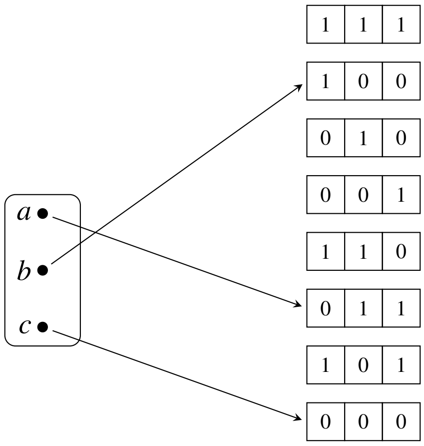

FIGURE 5. The relation $\{a, b, c\} \to \{1, 2, 3\}$ depicted as a function $\{a, b, c\} \to \{0, 1\}^3$ sending $a \mapsto \{2, 3\}$, $b \mapsto \{1\}$, $c \mapsto \emptyset$.

- Prove that the set $\mathcal{M}(U)$ can be identified with the set of all functions $U \to \mathbf{N}$.

Let $A \in \mathcal{M}(U)$ be a multiset. Define the **counting function** $\epsilon_A : U \to \mathbf{N}$ of A, mapping each element of U to the number of times that element occurs in A. We can use counting functions to define a 'generalised inclusion' relation $\leq$ for multisets. For $A, B \in \mathcal{M}(U)$, we write $A \leq B$ whenever for all $x \in U$, $\epsilon_A(x) \leq \epsilon_B(x)$.

- Prove or disprove that $\leq$ is a partial order relation on $\mathcal{M}(U)$.

Define the following operations on $\mathcal{M}(U)$:

- the **union** $A \curlyvee B$, with associated counting function
$$x \in U \mapsto \max\{\epsilon_A(x), \epsilon_B(x)\}$$
- the **intersection** $A \curlywedge B$, with associated counting function
$$x \in U \mapsto \min\{\epsilon_A(x), \epsilon_B(x)\}$$
- the **sum** $A \oplus B$, with associated counting function
$$x \in U \mapsto \epsilon_A(x) + \epsilon_B(x)$$
- the **difference** $A \ominus B$, with associated counting function
$$x \in U \mapsto \epsilon_A(x) - \epsilon_B(x)$$
if this number is nonnegative, and 0 otherwise (more formally, the counting function sends $x \in U$ to $\max\{0, \epsilon_A(x) - \epsilon_B(x)\}$).

EXERCISE OR.25$_\square$ Prove that $(\mathcal{M}(U), \curlywedge, \curlyvee)$ is a distributive lattice. Prove the **inclusion-exclusion** principle for multisets:
$$A \curlyvee B = (A \oplus B) \ominus (A \curlywedge B). \tag{1.10}$$

EXERCISE OR.26$_\square$ Count how many reflexive relations exist on a set with 7 elements. Same question, but with symmetric relations. Same question, but with transitive relations.

Exercise or.27$_\square$ Let P be the set of all sets $[n] = \{1, \dots, n\}$, for $0 \le n \le 7$, with the convention that $[0]$ is the empty set. Define the relation $[i] \le [j]$ on P if there is a function $[i] \to [j]$. Study the order-theoretic properties of the pair $(P, \le)$. Is $(P, \le)$ a poset? If not, what fails? Does P have a bottom element? A top element? Suprema for each subsets?...

Exercise or.28$_\square$ A relation R on X is an **equivalence relation** if it is reflexive, symmetric and transitive; count how many equivalence relations are there on a 4-elements set; count how many equivalence relations are there on a set with 17 elements (but do not ask me to do it at the exercise sessions!).

Denote $\mathsf{eRel}(X)$ the set of equivalence relations on X. Does the poset $(\mathsf{eRel}(X), \subseteq)$ have a top element? A bottom element?

Exercise or.29$_\square$ Which of these relations are equivalence relations on their respective domains?

- The relation R_1 defined on the set $\mathbf{Z}$ of integers, by the rule $(x, y) \in R_1$ if and only if $x - y$ is a multiple of 7.
- The relation R_2 defined on the set $\mathbf{N} = \{0, 1, 2, \dots\}$ of natural numbers, by the rule $(x, y) \in R$ if and only if the sum $x + y$ is a prime number.
- The relation R_2' defined on the set $\mathbf{N} = \{0, 1, 2, \dots\}$ of natural numbers, by the rule $(x, y) \in R_2'$ if and only if the product xy is a prime number.
- The relation R_3 defined on the set A^A of functions $f : A \to A$, by the rule $(f, g) \in R_3$ if and only if $f \circ g = g \circ f$ ($\circ$ denotes function composition).
- The relation R_4 defined on the set $\mathbf{R}$ of real numbers, by the rule $(x, y) \in R_4$ if and only if the difference $x - y$ is an integer.
- The relation R_5 defined on the set $\mathbf{Z} \times \mathbf{Z}$ of pairs of integers, asking that two pairs of integers $(x_1, x_2), (y_1, y_2)$ are in the relation R_5 if and only if $x_1 y_2 = x_2 y_1$.
- The relation R_5' defined on the set $\mathbf{Z} \times \mathbf{Z}^\times$ of pairs of integers *where the second component is not zero*, by the rule $(x_1, x_2), (y_1, y_2) \in R_5'$ if and only if $x_1 y_2 = x_2 y_1$.

Exercise or.30$_\square$ Given a relation R on X, the equivalence relation **generated** by R is the intersection of all equivalence relations on X containing R; we denote it $\bar{R}$. Show that $\bar{R}$ coincides with the subset of $X \times X$ defined 'recursively' as follows:

- $(x, x) \in \bar{R}$ for each $x \in X$;
- $(y, x) \in \bar{R}$ for each $(x, y) \in \bar{R}$;
- $(x, y) \in \bar{R}$ each time there are $z_0, z_1, \dots, z_n, z_{n+1} \in X$ such that $z_0 = x, z_{n+1} = y$ and $(z_i, z_{i+1}) \in R$ for each $i = 0, \dots, n$.

A **chain** in a poset $(P, \le)$ consists of a subset $C \subseteq P$ that is totally ordered as a subset of P. An **upper bound** for a subset $C \subseteq P$ of a poset is an element $m \in P$ such that $x \le m$ for every $x \in C$. A **maximal element** in $(P, \le)$ is an element $t \in P$ such that, if $t \le x$ for some $x \in P$, then $t = x$.

AC1) (**Zorn lemma**) Let $(X, \le)$ be a nonempty poset where every chain $C \subseteq X$ has an upper bound. Then, X admits a maximal element t.

AC2) (**Axiom of choice**) Let I be any set, and X_i a family of sets indexed by I. Let $X := \bigcup_{i\in I} X_i$. Then, there exists a **choice function** for the family $\{X_i\}$, i.e. a function $f : I \to X$ with the property that for each $i \in I$, $f(i) \in X_i$.

Exercise or.31 $\square$ Prove that Zorn lemma implies the Axiom of choice; prove (more difficult) that the Axiom of Choice implies Zorn lemma. You will have a hard time proving that the Axiom of choice 'just holds', meaning that it can be deduced from other statements than the Zorn lemma, in some possibly convoluted way. Do not try!

Exercise or.32 $\square$ Given a set X and an equivalence relation R on it, we call the **equivalence class** of an element x in X the set

$$[x]_R := \{y \in X \mid (x, y) \in R\} \tag{1.11}$$

Show that if $[x]_R \cap [y]_R \neq \varnothing$, then $[x]_R = [y]_R$ (two equivalence classes are either disjoint sets, or they coincide). The set X/R is the set of all equivalence classes of X defined by R:

$$X/R := \{[x]_R \mid x \in X\}. \tag{1.12}$$

Define a function $\pi_{/R} : X \to X/R : x \mapsto [x]_R$, called **projection to the quotient**.

Exercise or.33 $\square$ Let $f : A \to B$ be a function between sets A, B; the equivalence relation **induced by** f is the equivalence relation defined by $(a, a') \in R_f$ if and only if $fa = fa'$ (this means: a, a' have the same image under f). Describe the equivalence relations induced by the following functions:

- $f_1 : A \to B$ sending each element of A in a fixed $b_0 \in B$.
- $f_2 : \mathbf{Z} \to \mathbf{Z} : n \mapsto 7n$ multiplying an integer n by 7.
- $f_3 : \mathbf{R} \to \mathbf{R}$ taking the *floor* of a real number (the floor of $x \in \mathbf{R}$ is the greatest integer k such that $k \leq x$).
- $f_4 : \mathbf{Q} \to \mathbf{R}$ multiplying $q \in \mathbf{Q}$ for $\sqrt{27}$.
- $f_5 : A^A \to A^A : \varphi \mapsto \varphi^{\circ 7}$ that composes $\varphi : A \to A$ with itself seven times.
- $f_6 : \mathbf{N} \to \mathbf{N}$ sending m into m_0, where $m_k \dots m_1 m_0$ is the binary expansion of m in base 2.
- $f_7 : \mathbf{N} \to \mathbf{N}$ sending n into n^2.

Exercise or.34 $\square$ Let $f : X \to Y$ be a function between two sets; let R_f be the equivalence relation generated by f. Prove that f induced an injective function $\bar{f} : X/R_f \to Y$. Who is the image of $\bar{f}$? What can you deduce when f is surjective?

Exercise or.35 $\square$ A **partition** $\mathcal{E}$ of a set X consists of a family of pairwise disjoint subsets $E_i \subseteq X$ (this means that if $i \neq j$, $E_i \cap E_j = \varnothing$) such that $\bigcup E_i = X$. Show that every equivalence relation R on X defines a partition $\mathcal{E}(R)$ di X, and conversely, every partition $\mathcal{E}$ of X defines an equivalence relation $R(\mathcal{E})$ on X, in such a way that $\mathcal{E}(R(\mathcal{E})) = \mathcal{E}$ and $R(\mathcal{E}(R)) = R$.

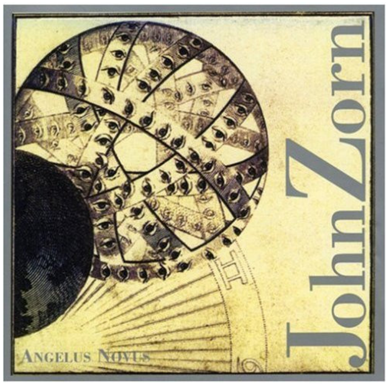

FIGURE 6. Zorn lemma, 1998

EXERCISE OR.36$_{\square}$ Describe X/R for the (equivalence) relations $R_1, \dots, R'_5$ in OR.29; for those that are not equivalence relations, decsribe $X/\bar{R}$. Describe $X/\bar{R}_{f_i}$ for $f_1, \dots, f_7$ in OR.33.

EXERCISE OR.37$_{\square}$ Use Zorn lemma to show that every infinite set X admits a partition in subsets X_α such that every X_α is countable. Use Zorn lemma to show that given *any* two sets X, Y, there exists either an injective function $f : X \to Y$, or an injective function $g : Y \to X$.

EXERCISE OR.38$_{\square}$ Define an equivalence relation Γ on the set $\mathsf{Rel}(X)$ of equivalence relations on X, positing that $(R, S) \in \Gamma$ if and only if there exists a bijection between X/R and X/S. How many elements does the quotient $\epsilon\mathsf{Rel}(X)/\Gamma$ have?

DEFINITION 1.9. Given posets P, Q, a pair of monotone maps

$$f : P \leftrightarrows Q : g \tag{1.13}$$

is called a *Galois connection* if for any two $x \in P, y \in Q$ the inequality $fx \leq y$ is true if and only if the inequality $x \leq gy$ is true.

EXERCISE OR.39$_{\square}$ Prove that every relation $R \in \mathsf{Rel}(A, B)$ defines a Galois connection between the set PA of subsets of A and the set PB of subsets of B: this means that there exists a pair of monotone functions

$$^R(-) : PA \to PB \qquad (-)^R : PB \to PA \tag{1.14}$$

such that $V \subseteq {}^R U$ if and only if $U \subseteq V^R$, for each $V \in PB$ and $U \in PA$.

EXERCISE OR.40$_{\square}$ On a graphical representation for relations. Show that a relation $R : X \to Y$ can be represented equivalently as follows: a pair of functions

$$X \xleftarrow{u} R \xrightarrow{v} Y \tag{1.15}$$

such that the function $R \to X \times Y$ defined as $r \mapsto (ur, vr) \in X \times Y$ is injective. In such a representation, a relation is denoted (R, u, v).

EXERCISE OR.41$_\square$ Define the **composition** of two relations $(R,u,v) : X \to Y$, $(S,w,t) : Y \to Z$ as

$$R \mathbin{⨟} S := \{(x,z) \mid \exists y \in Y.(x,y) \in R, (y,z) \in S\} \subseteq X \times Z. \tag{1.16}$$

EXERCISE OR.42$_\square$ Define a couple of functions h, k,

$$R \xleftarrow{h} R \mathbin{⨟} S \xrightarrow{k} S \tag{1.17}$$

so that $w(k(q) = v(h(q))$ for every $q \in R \mathbin{⨟} S$. This means that the function $wk = w \circ k$ coincides with the function $vh = v \circ h$; a graphical way to represent such a situation is to depict w, k, v, h as edges of a graph, in this case as a square

$$\begin{tikzcd} R \mathbin{⨟} S \arrow[r, "k"] \arrow[d, "h"'] & S \arrow[d, "w"] \\ R \arrow[r, "v"'] & Y \end{tikzcd} \tag{1.18}$$

and to declare that the square **commutes** when $wk = vh$.

EXERCISE OR.43$_\square$ Prove that given any other commutative square

$$\begin{tikzcd} E \arrow[r, "s"] \arrow[d, "r"'] & S \arrow[d, "w"] \\ R \arrow[r, "v"'] & Y \end{tikzcd} \tag{1.19}$$

There is a unique function $(r/s) : E \to R \mathbin{⨟} S$ with the property that $k \circ (r/s) = s$ and $h \circ (r/s) = r$. This is called the **universal property** of $R \mathbin{⨟} S$.

EXERCISE OR.44$_\square$ Let $\Delta_X = \{(x,x) \mid x \in X\}$. Prove that for every relation $(R,u,v) : X \to Y$ one has $\Delta_X \mathbin{⨟} R = R$ and $R \mathbin{⨟} \Delta_Y = R$. The relation Δ_X on a set X plays the role of **identity** for the composition operation on relations.

Let $(R,u,v) : X \to Y$, $(S,w,t) : Y \to Z$ be relations; prove that $(R \mathbin{⨟} S)^{\mathrm{op}} = S^{\mathrm{op}} \mathbin{⨟} R^{\mathrm{op}}$.

EXERCISE OR.45$_\square$ Let R be a relation on a set X; define the relation $\tilde{R}$ to be $R \cup R^{\mathrm{op}} \cup \Delta$, where Δ is the diagonal relation, as above. Prove that $\tilde{R}$ is the smallest reflexive and symmetric relation containing R.

EXERCISE OR.46$_\square$ Let X be a set and R a relation on X. Show that the transitive closure of R, i.e. the smallest transitive relation containing R, coincides with the set

$$\bigcup_{n=1}^{\infty} R^{⨟ n} := R \cup (R \mathbin{⨟} R) \cup (R \mathbin{⨟} R \mathbin{⨟} R) \cup \dots \tag{1.20}$$

Prove that th equivalence relation generated by R, as defined in OR.30, is the transitive closure of $\tilde{R}$ as defined above.

EXERCISE OR.47$_\square$ Let R, S be equivalence relations on a set X, such that $R \mathbin{⨟} S = S \mathbin{⨟} R$. Prove that $R \mathbin{⨟} S$ is an equivalence relation on X, and in fact it is the join $R \vee S$ of $\{R, S\}$ in the poset $(\epsilon\mathsf{Rel}(X), \subseteq)$.

Exercise or.48$_\square$ Let R, S, T, T' be relations on a set X. For each of the following items, prove it if they are true, or provide a counterexample if they are false.

- If $R \subseteq S$, then $R \mathbin{;} T \subseteq S \mathbin{;} T$ and $T \mathbin{;} R \subseteq T \mathbin{;} S$;
- $R \mathbin{;} (T \cap T') = (R \mathbin{;} T) \cap (R \mathbin{;} T')$ and $(T \cap T') \mathbin{;} R = (T \mathbin{;} R) \cap (T' \mathbin{;} R)$;
- $R \mathbin{;} (T \cup T') = (R \mathbin{;} T) \cup (R \mathbin{;} T')$ and $(T \cup T') \mathbin{;} R = (T \mathbin{;} R) \cup (T' \mathbin{;} R)$;
- $R \subseteq S$ if and only if $R^{\mathrm{op}} \subseteq S^{\mathrm{op}}$;
- $(T \cup T')^{\mathrm{op}} = T^{\mathrm{op}} \cup (T')^{\mathrm{op}}$ and $(T \cap T')^o p = T^{\mathrm{op}} \cap (T')^{\mathrm{op}}$;

Exercise or.49$_\square$ Prove **Szpilrajn extension theorem**: let $(P, \leq)$ be a (thin) poset; prove that there exists a relation $\preceq$ *extending* $\leq$, i.e. such that $x \leq y$ implies $x \preceq y$, which is also a (thin) total order, i.e. either $x \preceq y$ or $y \preceq x$. [Hint: this exercise can be described as a 'banal corollary of Zorn lemma'; what is nontrivial is to understand how to use AC1 on page 17]

Exercise or.50$_\square$ Let X, Y be posets; define a relation $\leq$ on the cartesian product $X \times Y$ by saying

$$(x, y) \leq (x', y') \iff x \leq x' \text{ in } X,\ y \leq y' \text{ in } Y. \tag{1.21}$$

Show that the projection functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are monotone maps, when $X \times Y$ is equipped with this order relation. Show that the diagonal map

$$d_X : X \longrightarrow X \times X \tag{1.22}$$

sending $x \in X$ into $(x, x) \in X \times X$ is monotone.

Now, let $(X, \vee, \wedge)$ be an algebraic lattice. Show that there are Galois connections

$$\_ \vee \_ :\ X \times X \rightleftarrows X\ : d_X \qquad d_X :\ \rightleftarrows X \times X\ : \_ \wedge \_ \tag{1.23}$$

if $\vee, \wedge : X \times X \to X$ are respectively the sup and inf operation on X.

Exercise or.51$_\square$ Let $f : X \to Y$ be a monotone map between posets; assume f fits in a Galois connection $f \dashv u$, where $u : Y \to X$. Prove that f preserves the bottom element of X, if it exists; prove that u preserves the top element of Y, if it exists. Prove that $f(\sup S) = \sup f(S)$ for every subset $S \subseteq X$ for which $\sup S$ exists; in particular, $f(x \vee x') = fx \vee fx'$ for every $x, x' \in X$ admitting a sup in X.

Exercise or.52$_\square$ Let $(P, \leq)$ be a poset; a **down-set** in P is a subset $S \subseteq P$ such that if $x \in S$ and $y \leq x$, then $y \in S$. Let DP be the set of all down-sets of P; for each $x \in P$, define the downset ${\downarrow}x$ **generated** by x as the set $\{y \in P \mid y \leq x\}$. Show that the map $x \mapsto {\downarrow}x$ is a monotone, injective map ${\downarrow}(-) : P \to DP$, i.e. show that

$$a \leq b \Rightarrow {\downarrow}a \leq {\downarrow}b. \tag{1.24}$$

Now let $(P, \leq)$ admit suprema for all down-sets; show that $S \mapsto \bigvee S$ defines a monotone map $DP \to P$, and show that

$$\bigvee S \leq x \iff S \subseteq {\downarrow}x \tag{1.25}$$

for every $S \in DP$ and $x \in P$.

EXERCISE OR.53□ Define ${\downarrow}U := \bigcup_{x\in U} {\downarrow}x$ for every $U \subseteq P$. Prove that U is a down-set if and only if ${\downarrow}U = U$.

EXERCISE OR.54□ In the same notation of OR.52, we call a lattice $(P, \leq)$ admitting all suprema **completely distributive** if there exists a monotone map ${\Downarrow}(-) : P \to DP$, such that

$$ {\Downarrow}a \subseteq S \iff a \leq \bigvee S \tag{1.26} $$

for all $a \in P$ and $S \in DP$. Show that the only possible definition for ${\Downarrow}a$ is as the set $\bigcap\{S \in DP \mid a \leq \bigvee S\}$. Define a relation $\ll$ on P as follows: $x \ll a$ if and only if $x \in {\Downarrow}a$. Prove that for any $a, b, x \in P$ one has

- if $x \ll a$ and $a \leq b$, then $x \ll b$;
- $a \leq \bigvee\{x \in P \mid x \ll a\}$.

EXERCISE OR.55□ Let $(P, \wedge, \vee)$ be a lattice. Prove that given a set I of indices and a family $\{S_i \mid i \in I\}$ of down-sets, the following equality holds:

$$ \bigvee \left(\bigcap_{i\in I} S_i \right) = \bigwedge_{i\in I} s_i \tag{1.27} $$

where $s_i := \bigvee S_i$. Let $\{A_i \mid i \in I\}$ be a family of subsets of P; prove that the above equality holds if and only if it holds fo downsets, i.e.

$$ \bigvee \left(\bigcap_{i\in I} {\downarrow}A_i \right) = \bigwedge_{i\in I} a_i \tag{1.28} $$

where $a_i := \bigvee A_i$.

EXERCISE OR.56□ On free distributive lattices on a set. Let A be a set; this series of exercises is intended to build the **free distributive lattice** on A, i.e. a lattice $(L[A], \wedge, \vee)$ enjoying the following properties:

- $(L[A], \wedge, \vee)$ is distributive, and there exists a function $\eta : A \to L[A]$;
- For every other distributive lattice $(X, \wedge, \vee)$, every function $f : A \to X$ can be extended to a unique lattice homomorphism $\bar{f} : (L[A], \wedge, \vee) \to (X, \wedge, \vee)$ such that $\bar{f} \circ \eta = f$.

DEFINITION 1.10. Given a set A, an **antichain**, or a *clutter*, or an *irredundant family* in A is a family of subsets $(E_\alpha \subseteq A \mid \alpha \in I)$ with the property that given $\alpha \neq \beta$ one has $E_\alpha \not\subseteq E_\beta$ (in words: none of the E_α contains one of the E_β as a subset). Define the set $L[A]$ to be the set of finite sets of finite antichains in A, i.e. the set of all

$$ \{X_1, \dots, X_n\} \tag{1.29} $$

where each X_i is a finite antichain in A.

EXERCISE OR.57□ Prove that given two antichains $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_m)$, one can obtain an antichain from $X \cup Y$ by 'removing repetitions': the set-theoretic union $\{X_1, \dots, X_n, Y_1, \dots, Y_m\}$ of the two antichains is not, in general, an antichain (find a counterexample for two antichains on $\{1, 2, 3, 4, 5\}$), but we can define a 'one-step reduction' operation $\rightsquigarrow$ by declaring that

$$ \{X_1, \dots, X_n, Y_1, \dots, Y_m\} \rightsquigarrow \{X_1, \dots, X_n, Y_1, \dots, \widehat{Y_j}, \dots, Y_m\} \tag{1.30} $$

if there exists an index i such that either $X_i \subseteq Y_j$ or $X_i \supseteq Y_j$. The relation of reduction $\rightsquigarrow^*$ now is defined to be the transitive closure of $\rightsquigarrow$; We denote the binary operation of union $(X, Y) \mapsto X \cup Y$, followed by complete reduction $\rightsquigarrow^*$ as $X \sharp Y$.

Given two elements $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_m)$ in $L[A]$, define the join of two elements in $L[A]$ as $X \vee Y := X \sharp Y$ and the meet $X \wedge Y := \{X_i \sharp Y_j \mid 1 \le i \le n, 1 \le j \le m\}$.

Exercise or.58$_\square$ Prove that these definitions equip $L[A]$ with the structure of a (distributive) lattice. Prove that $L[A]$ is the free distributive lattice on the set A. [Hint: interpret a generic element

$$\big\{\{a_{1,1}, \dots, a_{1,i_1}\}, \dots, \{a_{n,1} \dots, a_{n,i_n}\}\big\} \tag{1.31}$$

of $L[A]$ as the element $(a_{1,1} \wedge \cdots \wedge a_{1,i_1}) \vee \cdots \vee (a_{n,1} \wedge \cdots \wedge a_{n,i_n})$.]

Exercise or.59$_\square$ On filters. Let $(P, \le)$ be a poset. A nonempty subset A of P is called **down-directed** if for all $x, y \in A$ there exists a $z \in A$ such that $z \le x$ and $z \le y$. A subset A of P is called a **filter** if it is down-directed and up-closed; a filter is called **proper** if $A \ne P$. Show that if P has finite meets, a filter A is a subset of P such that

- for all $x, y \in A$, $x \wedge y \in A$;
- $\top \in A$;
- if $x \in A$ and $x \le y$, then $y \in A$.

Let $A \subseteq P$ be any down-directed subset; show that ${\uparrow}A = \bigcup_{x\in A} {\uparrow}x$, where ${\uparrow}x := \{y \in P \mid x \le y\}$ is a filter, called the filter **generated** by A. In particular, every ${\uparrow}x$ is a filter, the **principal filter** generated by x. Show that for every $A \subseteq X$, and every function $f : X \to Y$ one has $f({\uparrow}A) = {\uparrow}(fA)$.

Exercise or.60$_\square$ Let $f : X \to Y$ be a function; let $\mathfrak{a} \subseteq PX$ be a filter in the powerset of X (a common shorthand for this is to say that $\mathfrak{a}$ is a filter on X). Show that

$$\{B \subseteq Y \mid f^{\leftarrow}B \in \mathfrak{a}\} \tag{1.32}$$

defines a filter in PY, called the **direct image** filter on Y under f.

Let $\mathfrak{b} \subseteq PY$ be a filter in PY; show that

$$\{A \subseteq X \mid \exists B \in \mathfrak{b} : f^{\leftarrow}B \subseteq A\} \tag{1.33}$$

is a filter in PX, called the **inverse image filter** on X under f.

Exercise or.61$_\square$ Let X be a set, and $\mathfrak{A}$ a filter on the set of filters on X; define a filter $\sum \mathfrak{A}$ (called the **Kowalski sum** of $\mathfrak{A}$) on X as follows: $A \subseteq X$ is an element of $\sum \mathfrak{A}$ if and only if the set of filters on X that are also filters on A belongs to $\mathfrak{A}$. Prove that $\sum \mathfrak{A}$ is indeed a filter on X. Prove that $A \in \sum \mathfrak{A}$ if and only if the set of filters $\mathfrak{a}$ on X that contain A as an element belongs to $\mathfrak{A}$.

Definition 1.11. An **ultrafilter** $\mathfrak{x}$ on a set X is a maximal element within the set of proper filters on X, ordered by inclusion; i.e. $\mathfrak{x}$ is a proper filter on X such that, if $\mathfrak{a}$ is a proper filter on X containing $\mathfrak{x}$, then $\mathfrak{x} = \mathfrak{a}$.

Exercise OR.62$_\square$ Prove that the following conditions are equivalent for a filter $\mathfrak{x}$ on X.

- $\mathfrak{x}$ is an ultrafilter on X;
- for all $A, B \subseteq X$, if $A \cup B \in \mathfrak{x}$ then either $A \in \mathfrak{x}$ or $\mathfrak{b} \in \mathfrak{x}$;
- for every $A \subseteq X$, either $A \in \mathfrak{x}$ or $A^c = X \setminus A \in \mathfrak{x}$.

Exercise OR.63$_\square$ Let $f : X \to Y$ be a function between sets, and let $\mathfrak{x}$ be an ultrafilter on X; prove that the direct image of $\mathfrak{x}$ under f is again an ultrafilter. As a corollary, when $f : X \hookrightarrow Y$ is the inclusion of a subset, for every ultrafilter $\mathfrak{y}$ on Y such that $Y \in \mathfrak{y}$ the direct image filter $\mathfrak{y}|_X = \{U \cap X \mid U \in \mathfrak{y}\}$ is an ultrafilter on X.

Exercise OR.64$_\square$ Let X be a set. Prove that for every element $x \in X$, the principal filter ${}^{\uparrow}x$ is an ultrafilter on X.

Exercise OR.65$_\square$ Prove that if $\mathfrak{X}$ is an ultrafilter on the set of ultrafilters on X, then the Kowalski sum $\sum \mathfrak{X}$ is an ultrafilter on X.

Defining ultrafilters of different kinds requires the Axiom of Choice, in the form of Zorn lemma.

Exercise OR.66$_\square$ Use Zorn lemma to prove that given a set X, every proper filter $\mathfrak{a}$on X is contained in an ultrafilter. As a consequence, prove that for every filter $\mathfrak{b}$ and every filter $\mathfrak{a} \subset \mathfrak{b}$, there is an ultrafilter $\mathfrak{x}$ on X such that $\mathfrak{a} \subseteq \mathfrak{x}$ but $\mathfrak{b} \not\subseteq \mathfrak{x}$. As a consequence, prove that given a filter $\mathfrak{a}$ on a set X, $\mathfrak{a}$ is the intersection of all ultrafilters on X containing $\mathfrak{a}$.

CHAPTER 2

More on ordered sets

In the following, we will always denote $(L, \wedge, \vee)$ an algebraic lattice. Given two elements $a, b \in L$ we define the **interval** $[a, b]$ as the subset $\{y \mid a \leq y \leq b\}$ of L.

Exercise po.1 $\square$ Show that the set $\mathbf{I}(L)$ of intervals in L becomes a lattice, defining the lattice operations $\wedge, \vee$ on $\mathbf{I}(L)$.

Exercise po.2 $\square$ Show that the pair of functions

$$\_ \vee b : \ [a \wedge b, a] \rightleftarrows [b, a \vee b] \ : \_ \wedge a \tag{2.1}$$

defines a Galois connection; we say that a lattice L is **modular** when $\_ \vee b \dashv \_ \wedge a$ is an order-isomorphism between $[b, a \vee b]$ and $[a \wedge b, a]$.

Exercise po.3 $\square$ Show that the following conditions are equivalent for a lattice L:

- L is modular;
- for every $a, b, x \in L$, if $a \leq b$ then $(x \vee a) \wedge b \leq (x \wedge b) \vee a$ (and thus the equality holds);
- every interval $[a, b] \subseteq L$ has the following property: every $c \in [a, b]$ has at most one complement c' in $[a, b]$.

Exercise po.4 $\square$ Show that if L is complemented and modular, then every interval in L is also complemented.

Exercise po.5 $\square$ Prove that every distributive lattice (cf. Definition 1.8) is modular.

Definition 2.1. Let $I, J \in \mathbf{I}(L)$ be two intervals in the lattice L; we say that I and J are **similar**, and we write $I \asymp J$, if there exist $a, b \in L$ such that one of the

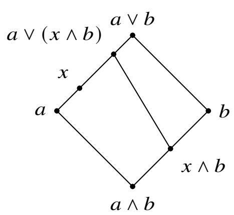

Figure 1. A minimal example of non-modular lattice, by construction.

intervals is $[a \wedge b, a]$ and the other is $[b, a \vee b]$. This defines a relation $\_ \asymp \_$ on $\mathbf{I}(L)$. Clearly, the $\_ \asymp \_$ relation is not transitive.

Exercise po.6$_\square$ Denote $\asymp^p$ the transitive closure of $\asymp$; is $\asymp^p$ an equivalence relation? We say that $I, J \in \mathbf{I}(L)$ are **projective** when $I \asymp^p J$. Show that projective intervals are order-isomorphic; is the converse true?

Denote respectively $0_L, 1_L$ the bottom and top element of the lattice L; two finite intervals with endpoints a, b, i.e. two chains

$$\underline{u} = \{a = u_0 \leq u_1 \leq \cdots \leq u_m = b\}$$
$$\underline{v} = \{a = v_0 \leq v_1 \leq \cdots \leq v_n = b\}$$

of elements of L are called **equivalent** if $m = n$ and there exists a permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ such that for evey $i = 1, n$ one has $[u_{i-1}, u_i] \asymp^p [v_{\pi i-1}, v_{\pi i}]$ (see po.6). We denote this situation as $\underline{u} \approx \underline{v}$.

Exercise po.7$_\square$ Show that $\approx$ is an equivalence relation.

A **refinement** of a finite interval $\underline{u}$ as above can be obtained by inserting further elements in the chain: more formally, given $\underline{u}, \underline{v}$ as above, we say that the interval $\underline{v}$ *refines* the interval $\underline{u}$ (and we write $\underline{u} \lhd \underline{v}$) if $n \geq m$ and $\{u_0, \dots, u_m\} \subseteq \{v_0, \dots, v_n\}$. Prove that refinement relation $\lhd$ is a partial order on $\mathbf{I}(L)$.

Exercise po.8$_\square$ Prove the **Schreier refinement lemma**: any two finite chains $\underline{u}, \underline{v}$ between the same pair of elements a, b in a modular lattice L admit equivalent refinements.

More formally: iven $\underline{u}, \underline{v}$ as above we can find intervals $\underline{x}, \underline{y}$ such that the following conditions are satisfied:

- $\underline{u} \lhd \underline{x}$;
- $\underline{v} \lhd \underline{y}$;
- $\underline{x} \approx \underline{y}$.

Definition 2.2. A **composition series** between $a, b \in L$ is a chain

$$a = u_0 \leq u_1 \leq \cdots \leq u_m = b \tag{2.2}$$

which has no refinement, except by introducing repititions of some of the given elements a_i. The integer m is the length of the chain.

Exercise po.9$_\square$ Prove the **Jordan-Hölder theorem** on composition series: any two composition series between the same pair of elements a, b in a modular lattice are equivalent.

Definition 2.3. A modular lattice L is of **finite length** if there is a composition chain between 0_L and 1_L; we define the *length* of L to be the length of such a composition chain (by the Jordan-Hölder theorem, this is well-defined).

Exercise po.10$_\square$ Prove that in a modular lattice of finite length, every chain

$$[a, b] = a = u_0 \leq u_1 \leq \cdots \leq u_m = b \tag{2.3}$$

can be refined to a composition series.

DEFINITION 2.4 (Noetherian and Artinian lattices). A lattice L is **Noetherian**, or *satisfies the ascending chain condition* if there is no infinite ascending sequence

$$a_0 < a_1 < a_2 < \dots \tag{2.4}$$

of distinct elements. Dually, L is called **Artinian**, or it *satisfies the descending chain condition* if there is no infinite descending sequence

$$\dots < a_2 < a_1 < a_0 \tag{2.5}$$

of distinct elements.

EXERCISE PO.11$_\square$ Prove that L is Noetherian (resp., Artinian) if and only if every nonempty subset S of L has a maximal (resp., minimal) element. [Assuming Noetherianity, you will need the Zorn lemma to prove the existence of a maximal element.]

EXERCISE PO.12$_\square$ Deduce from the previous exercise that a modular lattice is of finite length if and only if it is both noetherian and Artinian.

EXERCISE PO.13$_\square$ Let a be an element of a modular lattice L. Then L is Noetherian (resp., Artinian) if and only if both intervals $[0, a]$ and $[a, 1]$ are Noetherian (resp., Artinian).

EXERCISE PO.14$_\square$ Prove **Knaster-Tarski** fixpoint theorem: every monotone endofunction $f : L \to L$ of a complete lattice has at least a fixpoint. In fact, the set of fixpoints of f in L also forms a complete lattice, so that f has a *least* and a *greatest* fixpoint.

A converse of this theorem also holds: if every monotone function $f : L \to L$ on a lattice L has a fixpoint, then L is a complete lattice.

(Hint: define the set of prefix points of f as $P_f := \{x \in L \mid x \le fx\}$; show that $\bigvee P_f$ is a fixpoint of f. You can also argue in a similar way, using the set S_f of *postfix points* $S_f := \{x \in L \mid fx \le x\}$.)

DEFINITION 2.5. A totally ordered set W is called **well-ordered** if every non-empty subset $S \subseteq W$ admits a least element.

Well-ordered sets serve the purpose to classify order types: every well-ordered set is completely described by a certain distinguished element in its order-isomorphism class, an *ordinal number*. Roughly speaking, an ordinal number describes how you can line up the elements of a set so that the relation $\le$ is a well-order according to the definition above. Assuming the axiom of choice, it is possible to prove something quite strong: *every* set, no matter what is its internal structure, can be turned into a well-order. However, as it is customary with AoC-dependent theorems, there is no way to make this definition an explicit construction.

EXERCISE PO.15$_\square$ Let $(W, \le)$ be a well-ordered set, and let $f : W \to W$ be a monotone endofunction; show that for every $a \in W$, $a \le f(a)$.

EXERCISE PO.16$_\square$ Show that a well-ordered set is *rigid*: the only order-automorphism of a well-ordered set W is the identity.

Show that the isomorphisms between ordered sets are rigid: if there exists an order-isomorphism $f : V \to W$ between two well-ordered sets, then f is unique.

Exercise po.17$_\square$ Show that well-ordered sets are *trichotomous*: given two well-ordered sets V, W, exactly one of the following three cases holds:

- V is isomorphic to W; in this case, we write that the *order type* of V $o(V)$ is the same of the order type $o(W)$ of W.
- V is isomorphic to an initial segment of W; in this case, we write $o(V) \leq o(W)$.
- W is isomorphic to an initial segment of V; in this case, we write $o(W) \leq o(V)$.

Exercise po.18$_\square$ Assume that each well-ordered set W is assigned its ordinal $\alpha = o(W)$. Show that the assignment is well-defined and that $\leq$ totally orders the class Ord of all ordinals, and in fact, the pair $(\mathsf{Ord}, \leq)$ is a well-order...

... But it's not an ordinal: it lacks the property of being a *set*.

We want to have a more hands-on model of ordinals to work with: 'isomorphism classes of well-ordered sets' is a bit too elusive of a definition.

Definition 2.6. A set X is transitive if each element of X is also a subset of X:

$$x \in X \Rightarrow x \subseteq X. \tag{2.6}$$

Equivalently, X is transitive if $\bigcup X \subseteq X$, or $X \subseteq 2^X$.

A set X is an ordinal if it is transitive and well-ordered by the relation $\in$ (it's an $\in$-woset).

Historically, an ordinal is denoted with a lowercase Greek letter

$$\alpha, \beta, \gamma, \dots, \omega \tag{2.7}$$

or with 'decorated' versions thereof: α_0, γ', etc.

Exercise po.19$_\square$ Prove that

- The two definitions of ordinals do not confilict: given a well-ordered set W, there exists *exactly one* transitive and $\in$-woset X with an order isomorphism $W \cong X$;
- $0 := o(\varnothing)$ is an ordinal;
- if α is an ordinal, and $\beta \in \alpha$, then β is an ordinal;
- if α, β are ordinals and $\alpha \subsetneq \beta$ then $\alpha \in \beta$;
- if α, β are ordinals then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Exercise po.20$_\square$ Prove that

- for each ordinal α, $\alpha = \{\beta \mid \beta < \alpha\}$;
- if C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$;
- if X is a nonempty *set* of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$;
- for every ordinal α, the set $\alpha^+ := \alpha \cup \{\alpha\}$ is an ordinal, and $\alpha^+ = \inf\{\beta \mid \beta > \alpha\}$.

DEFINITION 2.7. A **successor ordinal** is an ordinal α such that there exists an ordinal β for which $\alpha = \beta^+$; α is then the *successor* of β, and it's usually written $\alpha = \beta + 1$.

A **limit ordinal** is an ordinal α such that $\alpha = \bigcup \alpha = \sup\{\beta \mid \beta < \alpha\}$ (in other words, a limit ordinal is such $\bigcup \alpha \subseteq \alpha$, *but also* $\alpha \subseteq \bigcup \alpha$).

We consider 0 to be a limit ordinal and define $0 = \sup \varnothing$.

EXERCISE PO.21 $\square$ Prove that α is a limit ordinal if and only if for every $\beta, \beta < \alpha$ implies $\beta + 1 < \alpha$.

The possibility to build limit ordinals relies on the axiom of infinity of ZF; in particular, one can

EXERCISE PO.22 $\square$ Prove that if a set X is inductive,[1] then $X \cap \mathsf{Ord}$ is also inductive, and the set $\omega = \bigcap\{X \mid X \text{ is inductive}\}$ is the least nonzero limit ordinal.

Stripped of its order, the ordinal ω is just 'the set of natural numbers' $\{0, 1, \dots\}$.

Without the axiom of infinity, the only ordinals to which we have access are $0 = \sup \varnothing$ an the *finite* ones $0^+ = 1 := \{\varnothing\}$, $2 = 1^+ = 0^{++} := \{\varnothing, \{\varnothing\}\}$, ..., $n + 1 := n^+$.

The power of ordinals lies in the fact that one can provide inductive definitions. A 'definition by transfinite recursion' usually takes the following form: let $\mathcal{K}$ be a class, then a function $h : \mathsf{Ord} \to \mathcal{K}$, called a **transfinite sequence**, is uniquely determined by

- the definition of the 'base' of the induction: a certain element h_0 of the class $\mathcal{K}$;
- the definition of the 'successor' step: a way to 'compute' $h_{\alpha+1}$ in terms of all $h_0, h_1, \dots, h_\alpha$;
- the definition of the 'limit' step: a way to 'compute' h_λ, when λ is a limit ordinal, in terms of all $h_\beta, \beta < \lambda$.

The most profitable way to employ such definitions is to define *arithmetic operations* on Ord: sum, product, and exponentiation.

Each such binary operation is specified by a recursion on the second argument, i.e. (for example for addition) there will be

- a base definition of what it means $\beta + 0$;
- a successor definition of what it means $\beta + (\alpha^+)$ in terms of $(\beta + 0, \beta + 1, \dots,)\beta + \alpha$;
- a limit definition of what it means $\beta + (\sup\{\theta \mid \theta < \lambda\})$.

DEFINITION 2.8 (Ordinal sum). The operation of ordinal sum is defined as follows: fix any ordinal β; then

- (base) $\beta + 0 := \beta$;
- (successor) $\beta + \alpha^+ := (\beta + \alpha)^+$;

[1] A set X is *inductive* if for every $x \in X$ also $x^+ = x \cup \{x\} \in X$. Since $\varnothing \in X$ for every X, a set X is inductive if it contains $\{\varnothing\}$, and thus also $\{\varnothing, \{\varnothing\}\}$, and thus also... The axioms of infinity says that there exists at leas an inductive set.

- (limit) $\beta + \bigcup\{\alpha \mid \alpha < \lambda\} := \bigcup\{\beta + \alpha \mid \alpha < \lambda\}$.

Exercise po.23$_\square$ Prove that the ordinal sum is associative; prove that it is commutative when restricted to finite ordinals $0, 1, 2 \dots$. Can you also prove that it is *not* commutative, because $\omega + 1$ is not order-isomorphic to $1 + \omega$? (Bonus points: prove that in fact $1 + \omega = \omega$ as ordinals.) Is it true that $0 + \alpha = \alpha$ for each ordinal α?[2]

Definition 2.9 (Ordinal product). The operation of ordinal product is defined as follows: fix any ordinal β; then

- (base) $\beta \cdot 0 := 0$;
- (successor) $\beta \cdot \alpha^+ := (\beta \cdot \alpha) + \beta$;
- (limit) $\beta \cdot \bigcup\{\alpha \mid \alpha < \lambda\} := \bigcup\{\beta \cdot \alpha \mid \alpha < \lambda\}$.

Exercise po.24$_\square$ Prove that the ordinal product is associative; prove that it is commutative when restricted to finite ordinals $0, 1, 2 \dots$. Can you also prove that it is *not* commutative, because $\omega \cdot 2$ is not order-isomorphic to $2 \cdot \omega$? (Bonus points: prove that in fact $2 \cdot \omega = \omega$ as ordinals). Is it true that $1 \cdot \alpha = \alpha$ for each ordinal α?

Definition 2.10 (Ordinal exponentiation). The operation of ordinal exponentiation is defined as follows: fix any ordinal β; then

- (base) $\beta^0 := 1$;
- (successor) $\beta^{(\alpha^+)} := (\beta^\alpha) \cdot \beta$;
- (limit) $\beta^{\bigcup\{\alpha \mid \alpha < \lambda\}} := \bigcup\{\beta^\alpha \mid \alpha < \lambda\}$.

(When it could be potentially confusing to adopt the superscript notation β^λ we might write $\exp(\beta, \lambda)$. So, $\exp(\beta, \alpha^+) = \exp(\beta, \alpha) \cdot \beta$ and $\exp(\beta, \sup\{\alpha \mid \alpha < \lambda\}) = \sup\{\exp(\beta, \alpha) \mid \alpha < \lambda\}$.)

Ordinal exponentiation is *not* associative, it's not commutative, it doesn't have a unit element.

Exercise po.25$_\square$ Prove the three identities

$$\beta^1 = \beta \qquad \beta^{\alpha+\gamma} = \beta^\alpha \cdot \beta^\gamma \qquad (\beta^\alpha)^\gamma = \beta^{\alpha\cdot\gamma} \tag{2.8}$$

for ordinal exponentiation, valid for all ordinals β, α, γ.

The operations of sum, product and exponentiation of ordinals can be succinctly constructed using just the successor function $\alpha \mapsto \alpha^+$ in a clever way. This is ultimately based on the fact that given the ordinal β, each function $\beta + \_, \beta \cdot \_, \exp(\beta, \_)$ can be regarded as a transfinite sequence $\mathsf{Ord} \to \mathsf{Ord}$.

Exercise po.26$_\square$ Let $h : \mathsf{Ord} \to \mathsf{Ord}$ be a function; define the α-th iterate of h by transfinite recursion as follows:

$$h^0 := \mathrm{id} \qquad h^{\alpha+1} = h^\alpha \circ h \qquad h^\lambda := \beta \mapsto \sup\{h^\alpha\beta \mid \alpha < \lambda\}. \tag{2.9}$$

Prove that

[2]Find an intuitive argument first, and then formalise the fact that $1 + \omega = \sup\{1 + \alpha \mid \alpha < \omega\}$ equals ω. Watch this video watch?v=YApat9UmUNg for inspiration.

- $\beta + \alpha = (\_ + 1)^\alpha \beta$;
- $\beta \cdot \alpha = (\_ + \beta)^\alpha 0$;
- $\exp(\beta, \alpha) = (\_ \cdot \beta)^\alpha 1$.

The point is that now we can apply the same recursive definition for the iterates of the exp function, building ω^ω, ω^{ω^ω}, $\omega^{\omega^{\omega^\omega}}$... more formally, we define a transfinite sequence

$$h_0 := \omega \qquad h_{\alpha+1} := \omega^{h_\alpha} \qquad h_\lambda := \sup\{h_\beta \mid \beta < \lambda\} \tag{2.10}$$

Now, what is h_ω *exactly*? It is a tower of ω's that is ω steps high.

Exercise po.27$\square$ Prove that h_ω is a fixpoint for the function $x \mapsto \omega^x$, or in other words a solution to the equation

$$\omega^x = x \tag{2.11}$$

in Ord.

The ordinal h_ω, that is more often written ϵ_0, is then equal to

$$\sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \dots\}$$

. The notation suggests that there exists an entire hierarchy of greater ϵ's, $\epsilon_1, \epsilon_2, \dots$ where

$$\epsilon_1 = \sup\{\epsilon_0 + 1, \omega^{\epsilon_0+1}, \omega^{\omega^{\epsilon_0+1}}, \omega^{\omega^{\omega^{\epsilon_0+1}}}, \dots\} \tag{2.12}$$

and more generally

$$\epsilon_{\alpha+1} = \sup\{\epsilon_\alpha + 1, \omega^{\epsilon_\alpha+1}, \omega^{\omega^{\epsilon_\alpha+1}}, \dots\} \tag{2.13}$$

Exercise po.28$\square$ Prove that

- $\epsilon_1 = \sup\{0, 1, \epsilon_0, {\epsilon_0}^{\epsilon_0}, {\epsilon_0}^{{\epsilon_0}^{\epsilon_0}}, \dots\}$;
- $\epsilon_{\alpha+1} = \sup\{0, 1, \epsilon_\alpha, \epsilon_\alpha^{\epsilon_\alpha}, \epsilon_\alpha^{\epsilon_\alpha^{\epsilon_\alpha}}, \dots\}$;
- $\epsilon_\omega = \sup\{\epsilon_0, \epsilon_1, \epsilon_2, \dots\}$.

It is now possible to define

$$\epsilon_{\omega+1}, \epsilon_{\omega+2}, \dots, \epsilon_{\omega+\omega} = \epsilon_{2\cdot\omega}, \dots, \epsilon_{\omega\cdot\omega} = \epsilon_{\omega^2}, \epsilon_{\omega^3}, \dots, \epsilon_{\omega^\omega}, \dots$$

up to ϵ_{ϵ_0}; what next?

Isn't that obvious? By transfinite recursion, $h_0 = \epsilon_0$, and $h_{\alpha+1} = \epsilon_{h_\alpha}$, whose smallest fixpoint will be an infinitely descending tower of ϵ's, $\epsilon_{\epsilon_{\epsilon_{\dots}}}$, with ω nested subscripts.

We could call this ordinal γ_0, and build $\gamma_1, \gamma_2, \dots, \gamma_{\gamma_0}, \dots, \gamma_{\gamma_{\gamma_0}}, \dots$, up to the smallest fixpoint of the function $h_0 = \gamma_0$, $h_{\alpha+1} = \gamma_{h_\alpha}$. However, the Greek alphabet will sooner or later fit badly in this picture and turn out to be a very unwieldy choice of notation at some point. A more systematic approach to this notational problem is given by the *Veblen hierarchy* of fixpoint-counting functions, but this is a longer story than it is worth explaining here.

A *normal function* is a class function $f : \mathsf{Ord} \to \mathsf{Ord}$ such that:

- f is strictly monotone: if $\alpha < \beta$, then $f(\alpha) < f(\beta)$;
- f is *continuous*: for every limit ordinal λ, $f(\lambda) = \sup\{f(\alpha) : \alpha < \lambda\}$.

Exercise po.29$_\square$

- Prove that the following operations are normal functions: $x \mapsto \alpha + x, x \mapsto \alpha \cdot x, x \mapsto \beta^x$ for every $\alpha, \beta \in \mathsf{Ord}, \beta > 1$.
- Prove that the composition of two normal functions is normal.
- Prove the **Veblen fixpoint lemma** for normal functions: if $f : \mathsf{Ord} \to \mathsf{Ord}$ is normal, then it has a fixpoint, i.e., for some ordinal α, $f(\alpha) = \alpha$, and in fact the class $\mathsf{Fix}(f) = \{\alpha \in \mathsf{Ord} \mid f(\alpha) = \alpha\}$ is unbounded and *closed*: if $A \subseteq \mathsf{Fix}(f)$ then $\sup A$ exists in $\mathsf{Fix}(f)$.

Given a poset $(P, \leq)$ and a subset $A \subseteq P$, consider the set ${}^{\uparrow}A = \{x \in P \mid \forall a \in A, a \leq x\}$ of all upper bounds of A, and the set ${}^{\downarrow}A = \{x \in P \mid \forall a \in A, x \leq a\}$ of all lower bounds of A. The **Isbell envelope** $\mathsf{Isb}(P)$ of P consists of the set all subsets $A \subseteq P$ with the property that ${}^{\downarrow}({}^{\uparrow}A) = A$, ordered by inclusion. An element $A \in \mathsf{Isb}(P)$ will be called *Isbell-closed*.

Exercise po.30$_\square$ Show that ${}^{\downarrow}(-) \dashv {}^{\uparrow}(-)$ is a Galois connection $(2^P, \supseteq) \to (2^P, \subseteq)$; this means that $U \subseteq {}^{\downarrow}V$ if and only if $V \subseteq {}^{\uparrow}U$; deduce that the following inequalities hold:

- for every $A \subseteq P$, $A \subseteq {}^{\downarrow}({}^{\uparrow}A)$;
- ${}^{\uparrow}({}^{\downarrow}({}^{\uparrow}A)) = {}^{\uparrow}A$.

Exercise po.31$_\square$ Define a monotone function $\jmath : P \to \mathsf{Isb}(P)$ sending $x \in P$ to the principal ideal ${}^{\downarrow}x$ (show that this is well-defined, i.e. that ${}^{\downarrow}x$ is Isbell-closed, and that $\jmath$ is monotone).

Exercise po.32$_\square$ Show that the Isbell envelope of P is order-isomorphic to the poset of **cuts** in P: a cut in P is a pair (A, B) of subsets of P such that ${}^{\uparrow}A = B$ and ${}^{\downarrow}B = A$. (Show that, if (A, B) is a cut, A is Isbell-closed, an conversely if A is Isbell-closed...), and for two cuts $(A, B), (A', B')$ we define $(A, B) \leq (A', B')$ if and only if $A \subseteq A'$. (Show that this is in turn equivalent to $B' \subseteq B$).

Exercise po.33$_\square$ Show that $\mathsf{Isb}(P)$ is a complete lattice equipped with an injective monotone function $P \hookrightarrow \mathsf{Isb}(P)$; show the *universal property* of the Isbell envelope: given an injective monotone function $\eta : P \to D$, with codomain a complete lattice, there exists a monotone embedding $\bar{\eta} : \mathsf{Isb}(P) \hookrightarrow D$ such that the triangle

$$\begin{tikzcd} P \arrow[rr, "\eta"] \arrow[dr, "\jmath"'] & & D \\ & \mathsf{Isb}(P) \arrow[ur, "\bar{\eta}"'] & \end{tikzcd} \tag{2.14}$$

is commutative.

Exercise po.34$_\square$ (Difficult.) Assume the poset $(P, \leq)$ has a certain Hasse diagram H_P; devise a method to find the Hasse diagram of $\mathsf{Isb}(P)$.

Definition 2.11. A **locally finite** poset P is a poset $(P, \leq)$ such that each element $[a, b] \in \mathbf{I}(P)$ of the interval poset of P is a finite set.

Exercise po.35□ Let P be a locally finite poset. Define the **incidence algebra** JP of a poset P to be the set of all functions $f : \mathbf{I}(P) \to \mathbf{C}$, assigning to each interval $[a,b] \in \mathbf{I}(P)$ a complex number $f_{[a,b]}$. On this set we define the **convolution** operation as follows:

$$f * g : [a,b] \mapsto \sum_{a \le x \le b} f_{[a,x]} g_{[x,b]}. \tag{2.15}$$

Show that the incidence algebra JP of P is a monoid, wen it is equipped with the convolution product and when the identity element is the 'delta' function $\delta : [a,b] \mapsto 1$ if $a = b$ and 0 otherwise.

Exercise po.36□ The *zeta function* of an incidence algebra JP is the constant function $\zeta(a,b) = 1$ for every nonempty interval $[a,b]$. Prove that ζ is an invertible element of JP (with respect to convolution); the inverse is the *Möbius function* of P, defined as follows:

$$\mu(x,y) = \begin{cases} 1 & x = y \\ \sum_{x \le z < y} \mu(x,z) & x < y \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

Prove that indeed $\zeta * \mu = \delta = \mu * \zeta$.

Exercise po.37□ Find an explicit expression for $\zeta * \zeta$. What about $\zeta * \zeta * \zeta$? Generalise.

Consider the poset $(\mathbf{N}, \_ \mid \_)$ of positive integers partially ordered by divisibility. The *reduced* incidence algebra consists of functions $f(a,b)$ that are invariant under multiplication, i.e. $f(ka,kb) = f(a,b)$ for all $k \ge 1$.

For a multiplicative invariant function, the value $f(a,b)$ depends only on b/a, so a natural basis consists of *invariant delta functions* δ_n defined by $\delta_n(a,b) = 1$ if $b/a = n$ and 0 otherwise: any invariant function can be written $f = \sum_{n \ge 0} f(1,n) \cdot \delta_n$.

Exercise po.38□ Show that the convolution of two invariant delta functions is still an invariant delta function.

To every element of the reduced incidence algebra we associate the *Dirichlet series* $\kappa_f := \sum_{n \ge 1} \frac{f(1,n)}{n^s}$.

The zeta function ζ belongs to the reduced incidence algebra and its associated Dirichlet series corresponds to the so-called *Riemann zeta function*

$$\zeta(s) = \textstyle\sum_{n \ge 1} \frac{1}{n^s}. \tag{2.17}$$

Exercise po.39□

- Show that $0 = \zeta(0) = \zeta(-2) = \zeta(-4) = \cdots = \zeta(-2k)$ for all $k \ge 0$.
- (difficult) Show that all other zeros of ζ belong to the set $\frac{1}{2} + i\mathbf{R} = \{\frac{1}{2} + it \mid t \in \mathbf{R}\}$.

CHAPTER 3

Semigroups, monoids

Exercise sm.1 □ If $a, b \in \mathbf{R}^+$ are strictly positive real numbers, the quotient a/b is still strictly positive. Is the set $(\mathbf{R}^+, /)$ endowed with the operation $(a, b) \mapsto a/b$ a semigroup? Is it commutative?

Exercise sm.2 □ Define a binary operation on the set of natural numbers as follows:

$$a \circ b := a + b + ab \tag{3.1}$$

Show that $(\mathbf{N}, \circ)$ is a commutative semigroup.

Exercise sm.3 □ Let $(S, \cdot)$ be a monoid, and X any set. Define a binary operation $*$ on the set S^X of all functions $X \to S$ as follows:

$$(f, g) \mapsto f * g : x \mapsto f(x) \cdot g(x) \tag{3.2}$$

Show that $(S^X, *)$ is a monoid, and that $(S^X, *)$ is commutative if $(S, \cdot)$ is. Is the converse implication true (if $(S^X, *)$ is commutative, $(S, \cdot)$ is commutative)?

Exercise sm.4 □ Let S be a *finite* set, and consider the monoid S^S of all functions $f : S \to S$. Prove that f is invertible if and only if it is an injective function, if and only if it is a surjective function. Find a counterexample when S is not finite.

Exercise sm.5 □ Consider the set $S = \mathbf{R}^\times \times \mathbf{N}$, where $\mathbf{R}^\times = \mathbf{R} \setminus \{0\}$; define a binary operation on S as follows:

$$(a, n)(b, m) := (ab^n, nm) \tag{3.3}$$

Show that, with this definition, S is a semigroup. Is it a monoid? Is it commutative?

Exercise sm.6 □ Let A^A be the monoid of endofunctions of a set A; count how many elements there are in A^A if $A = \{1, 2, 3, 4, 5\}$. Let $f(a) = \min\{a^2, 5\}$, and consider the cylic monoid $N = \langle f \rangle$ generated by f; how many elements does N have? Which ones?

Exercise sm.7 □ Let $\mathbf{Z} \times \mathbf{Z}$ be the set of pairs of integers (m, n). Define a monoid operation

$$(p, q)(r, s) := (pr - qs, ps + qr) \tag{3.4}$$

prove that it is indeed a monoid operation and that the function

$$v : \mathbf{Z} \times \mathbf{Z} \to \mathbf{N} : (p, q) \mapsto p^2 + q^2 \tag{3.5}$$

is a monoid homomorphism.

The relation of *congruence modulo an integer* is arguably the most important kind of equivalence relation in all Mathematics. Let $\mathbf{Z}$ be the set of integers

$\{\dots, -2, -1, 0, 1, 2, \dots\}$, and fix an integer $n \in \mathbf{Z}_{\geq 1}$. Define an equivalence relation $\_ \equiv_n \_$ on $\mathbf{Z}$ as follows: $a \equiv_n b$ if and only if $a - b$ is a multiple of n.

Exercise sm.8$_\square$ Show that $\equiv_n$ is an equivalence relation and, in fact, a *congruence* on the set of integers, namely that if $a \equiv_n b$, then for every $c \in \mathbf{Z}$ $a+c \equiv_n c+b$ and $c \cdot a \equiv_n c \cdot b$.

Exercise sm.9$_\square$ Consider the set $\mathbf{Z}/n := \mathbf{Z}/{\equiv_n}$ of equivalence classes modulo $\equiv_n$. Show that $\mathbf{Z}/n$ is a finite set, with exactly n elements $\{[0], [1], \dots, [n-1]\}$. Define a *sum* operation on $\mathbf{Z}/n$,

$$[a] + [b] = [a + b] \tag{3.6}$$

and a product operation

$$[a] \cdot [b] = [a \cdot b] \tag{3.7}$$

Show that $+, \cdot$ are well-defined, and that $(\mathbf{Z}, +), (\mathbf{Z}, \cdot)$ are commutative monoids. Show that, moreover, $(\mathbf{Z}, +)$ is an Abelian group.

Exercise sm.10$_\square$ Let $f : M \to N$ a surjective monoid homomorphism. Prove that if M is cyclic, so is N.

Exercise sm.11$_\square$ Let $\mathbf{R}^+ = \{x \in \mathbf{R} \mid x \geq 0\}$. Prove that $(\mathbf{R}^+, \cdot, 1)$ is a submonoid of $(\mathbf{R}, \cdot, 1)$. Prove that the function $\log_2 : \mathbf{R}^+ \to \mathbf{R}$ sending x into $\log_2 x$ is a monoid isomorphism.

Exercise sm.12$_\square$ Let $f : M \to M'$ be a monoid homomorphism. Then,

- prove that if N is a submonoid of M, its image $f(N)$ is a submonoid of M';
- prove that if N' is a submonoid of M', then $f^{\leftarrow}(N')$ is a submonoid of M.

Exercise sm.13$_\square$ Let M be the monoid $(\mathbf{N}, +, 0) \times (\mathbf{N}, +, 0)$, where

$$(a, b) + (c, d) = (a + c, b + d) \tag{3.8}$$

and where the unit is $(0, 0)$. Let $a, b \in \mathbf{N}$ and define the function

$$f_{a,b} : M \to \mathbf{N} : (x, y) \mapsto a^x b^y \in M. \tag{3.9}$$

Prove that $f_{a,b}$ is a homomorphism $M \to (\mathbf{N}, \cdot, 1)$. Prove that if $a > 1$ and $b = a^2$, $f_{a,b}$ is not injective. Prove that if a, b are different prime numbers, then f is injective.

Exercise sm.14$_\square$ (Cayley's theorem for monoids.) Prove that every monoid is isomorphic to a submonoid of $(X^X, \circ)$ for a suitable set X [hint: take X equal to the monoid M, and for each $a \in M$ define the function $f_a : x \mapsto ax$.].

Exercise sm.15$_\square$ Let $S = \{-1, 0\}$ regarded as a subset of $\mathbf{R}$. Describe the submonoid $\langle S \rangle$ of $(\mathbf{R}, \cdot)$. Prove that there exists a unique endomorphism $\varphi : (\mathbf{R}, \cdot) \to (\mathbf{R}, \cdot)$ with the property that $\varphi(0) = 0$ and $\varphi(a) = -1$ for every $a < 0$. Prove that the image of such φ is exactly S.

Exercise sm.16$_\square$ Let X be a set and consider the monoid $(PX, \cup)$ of subsets of X, where the monoid operation is given by the union. Let $S = \{\{a\} \mid a \in X\}$ be the subset of PX whose elements are singletons. Describe the submonoid $\langle S\rangle$ in $(PX, \cup)$.

Exercise sm.17$_\square$ In the same notation as above, describe the cyclic submonoid $\langle a\rangle \leq (PX, \cup)$ for a given $a \in X$.

Exercise sm.18$_\square$ Recall that a *group* is a monoid $(M, \cdot, 1_M)$ such that every element is invertible (this means that every element $x \in M$ has an *inverse* x^{-1}, such that $x \cdot x^{-1} = 1_M = x^{-1} \cdot x$). Prove that a semigroup $(S, \cdot)$ is a group if and only if for every $a \in S$ we have the equalities

$$aS := \{as \mid s \in S\} = S = Sa := \{sa \mid s \in S\}. \tag{3.10}$$

The notation aS is a particular case, and shorthand, for the following more general definition: let A, B be subsets of a semigroup S; denote as $A \cdot B$ or just AB the set $\{a \cdot b \mid a \in A, b \in B\}$; in particular, aB is a shorthand for $\{a\} \cdot B$.

This notation clearly extends to the 'product' of n subsets $A_1, \dots A_n \subseteq S$:

$$A_1 \cdot \dots \cdot A_n := \{a_1 \dots a_n \mid a_i \in A_i, 1 \leq i \leq n\}. \tag{3.11}$$

Exercise sm.19$_\square$ Let $(S, \cdot)$ be a finite semigroup satisfying both the left and right cancellation laws: if $xy = xz$ then $y = z$, and if $yx = zx$, then $y = z$. Prove that S is a group.

Exercise sm.20$_\square$ An idempotent element of a monoid $(M, \cdot, 1_M)$ is an element $e \in M$ such that $e \cdot e = e$. Prove that a finite monoid is a group if and only if it has a unique idempotent element (and that element is the identity 1_M).

Exercise sm.21$_\square$ Let $(S, \cdot)$ be a finite semigroup, and $a \in S$; show that there exists a smallest positive integer $n \in \mathbf{N}$, called the *index* of a, such that $a^n = a^{n+d}$ for some $d > 0$. The smallest possible choice of d is called the *period* of a. Show that $s^c = s^{n+rd}$ for every $r \geq 0$; show that $s^p = s^q$ if and only if $p = q < n$, or $p, q \geq n$ and $p \equiv_d q$.

Exercise sm.22$_\square$ In the same notation above, assume S is a finite semigroup, say of cardinality n. Let S^n be the set of products $s_1 \dots s_n$ of n elements of S. Prove the *pumping lemma*: the set S^n equals the set

$$S \cdot E_S \cdot S = \{s \cdot e \cdot s' \mid s, e, s' \in S, ee = e\} \tag{3.12}$$

(in words: every element of S^n can be written as a product ses', where $s, e, s' \in S$ and e is an idempotent).

Exercise sm.23$_\square$ Let S, T be finite semigroups; let $f : S \to T$ be a surjective homomorphism; prove that the image of the set of idempotents of S, $E_S = \{e \in S \mid ee = e\}$ under f equals the set E_T of idempotents of T.

Exercise sm.24$_\square$ Let $(M, \star)$ be a monoid; define $x \sim y$ if there exists $n \in \mathbf{N}$ such that $x^n = y^n$ in M. Show that $\sim$ is an equivalence relation on M, and that if M is commutative and $x \sim y$, then $a \star x \sim a \star y$ for all $a \in M$. Describe the equivalence class of 1_M.

Exercise sm.25$_\square$ Let A be a set, and $(A^A, \circ)$ the monoid of all functions $f : A \to A$. Fix a subset $B \subseteq A$. Show that the set

$$S(B) := \{f : A \to A \mid f(B) \subseteq B\} \tag{3.13}$$

is a submonoid of A^A. Prove that the function $\psi : S(B) \to B^B$ sending $f : A \to A$ to its restriction to B, $f|_B : B \to B$ is a surjective monoid homomorphism. Define an equivalence relation on $S(B)$ saying that $f \sim g$ if $f(b) = g(b)$ for every $b \in B$; prove that $\sim$ is precisely the equivalence relation induced by ψ.

Exercise sm.26$_\square$ Define an equivalence relation on the set $\mathbf{N} \times \mathbf{N}$ as follows:

$$(p, q) \sim (r, s) \iff p + s = q + r. \tag{3.14}$$

Define the sum on $\mathbf{N} \times \mathbf{N}$ as follows:

$$(p, q) + (r, s) = (p + r, q + s); \tag{3.15}$$

define the product

$$(p, q)(r, s) = (pr + qs, ps + qr). \tag{3.16}$$

Prove that

- $\sim$ is an equivalence relation, and a congruence on $\mathbf{N} \times \mathbf{N}$;
- the set $(\mathbf{N} \times \mathbf{N}, +)$ is a monoid;
- the set $(\mathbf{N} \times \mathbf{N}, \cdot)$ is a monoid;
- the *distributive property* holds for every $(a, b), (p, q), (r, s) \in \mathbf{N} \times \mathbf{N}$:

$$(a, b) \cdot ((p, q) + (r, s)) = (a, b)(p, q) + (a, b)(r, s). \tag{3.17}$$

- $\sim$ is the equivalence relation generated by $f : \mathbf{N}\times\mathbf{N} \to \mathbf{Z} : (a, b) \mapsto a-b$.

Exercise sm.27$_\square$ In the same notation of the previous exercise, show that $f : \mathbf{N} \times \mathbf{N} \to \mathbf{Z} : (a, b) \mapsto a - b$ induces an isomorphism $\mathbf{Z} \cong (\mathbf{N} \times \mathbf{N})/\!\sim$.

An important role in monoid theory is played by the notion of *ideal* and the closely related notion of *Green's relations*. In the following, let M be a finite monoid.

A *left ideal* (respectively, *right ideal*) of M is a nonempty subset $I \subseteq M$ such that $MI \subseteq I$ (respectively, $IM \subseteq I$). A *two-sided ideal*, or *ideal* is a nonempty subset $I \subseteq M$ such that $MIM \subseteq I$. We write $I \trianglelefteq M$ to denote the fact that $I \subseteq M$ and I is an ideal. The set $(\mathsf{Idl}(M), \trianglelefteq)$ is a poset.

Exercise sm.28$_\square$ If $I \trianglelefteq M$ is an ideal of a finite monoid M, prove that I contains at least an idempotent element $e = ee$.

Exercise sm.29$_\square$ Let M be a finite monoid; then it has a finite number of ideals $I_1, \dots, I_r$; prove that the product $I_1 I_2 \dots I_r$ is still an ideal of M, and that it is contained in every other ideal. Consequently, each finite monoid M has a unique *minimal* ideal $I_{\min}$.

Let $m \in M$ be an element of a finite monoid M. The ideals Mm, mM and MmM are respectively called the principal left, principal right, and principal two-sided ideal generated by m.

Exercise sm.30 $\square$ Define

$$I[m] := \{x \in M \mid m \notin MxM\}; \tag{3.18}$$

prove that if $I[m] \neq \emptyset$, then it is an ideal of M; prove that $I[m]$ is empty if and only if it is contained in $I_{\min}$.

Exercise sm.31 $\square$ Let M be a finite monoid. Prove that the following defines three equivalence relations on M:

- $x \equiv y \pmod{\mathfrak{j}}$ if and only if $MxM = MyM$;
- $x \equiv y \pmod{\mathfrak{l}}$ if and only if $Mx = My$;
- $x \equiv y \pmod{\mathfrak{r}}$ if and only if $xM = yM$.

(Writing 'mod $\mathfrak{j}$' is just a slick shorthand: take for example the $\mathfrak{j}$ relation; we write $x \equiv y \pmod{\mathfrak{j}}$ or $x \equiv_{\mathfrak{j}} y$ to denote that $(x, y) \in \mathfrak{j} \subseteq M \times M$.) These are called respectively the *two-sided*, *left* and *right* Green relations on M. We say that x, y are *two-sided Green equivalent*, or $\mathfrak{j}$-equivalent, if $x \equiv y \pmod{\mathfrak{j}}$. Similarly, we say that x, y are left Green equivalent, or $\mathfrak{l}$-equivalent, if $x \equiv y \pmod{\mathfrak{l}}$, and right Green equivalent, or $\mathfrak{r}$-equivalent, if $x \equiv y \pmod{\mathfrak{r}}$.

Exercise sm.32 $\square$ Let $a, b, c \in M$ be elements of a finite monoid. Prove that if $a \equiv_{\mathfrak{r}} b$, then $ca \equiv_{\mathfrak{r}} cb$ and if $a \equiv_{\mathfrak{l}} b$, then $ac \equiv_{\mathfrak{l}} bc$; is it also true that if $a \equiv_{\mathfrak{r}} b$, then $ac \equiv_{\mathfrak{l}} bc$?

Exercise sm.33 $\square$ Let M be a finite monoid. Prove that the relation $\mathfrak{j}$ is the join of $\mathfrak{r}, \mathfrak{l}$ in the poset of equivalence relations on M.

Exercise sm.34 $\square$ Let k be a field and $n \geq 1$ an integer; consider the monoid $M_n(k)$ of $n \times n$ matrices with entries in k. Show that two matrices $A, B \in M_n(k)$ are $\mathfrak{j}$-equivalent if and only if they have the same rank.

Exercise sm.35 $\square$ Let A^A be the monoid of endofunctions of a finite set A; show that $f, g \in A^A$ are $\mathfrak{j}$-equivalent if and only if their images have the same cardinality; $f, g \in A^A$ are $\mathfrak{l}$-equivalent if and only if the associated equivalence relations R_f, R_g are equal; $f, g \in A^A$ are $\mathfrak{r}$-equivalent if and only if they have the same image.

Exercise sm.36 $\square$ A monoid M is called $\mathfrak{r}$*-trivial* if $mM = nM$ implies $m = n$, or in other words, if the relation $\mathfrak{r}$ reduces to the identity Δ_M; similarly we define a $\mathfrak{l}$-trivial monoid, and a $\mathfrak{j}$-trivial monoid M (in the latter case, we mean that if $MmM = MnM$, then $m = n$. Prove that a monoid is $\mathfrak{j}$-trivial if and only if it is both $\mathfrak{r}$-trivial and $\mathfrak{l}$-trivial.

Exercise sm.37 $\square$ Implement *Light associativity test* in your favourite programming language: given a finite set S, a sufficient condition for a binary operation $\star : S \times S \to S$ to be associative is that given any element $y \in S$, the following two tables coincide:

- the 'matrix' $L_1(y)$ constructed from an enumeration $S = \{x_1, \dots, x_n\}$ whose entry (i, j) is $x_i \star (y \star x_j) \in S$;
- the 'matrix' $L_2(y)$, constructed from the same enumeration, whose entry (i, j) is $(x_i \star y) \star x_j$.

The operation $\star$ is associative if and only if for every $y \in S$, $L_1(y) = L_2(y)$.

DEFINITION 3.1. A **partially ordered monoid** is a monoid $(M, \cdot, 1)$ equipped with a partial order $\leq$ that is *compatible* with the monoid operation; this means that for each $a, b, c \in M$, if $b \leq c$ then $ab \leq ac$ and $ba \leq ca$.

A partially ordered monoid $(M, \leq, \cdot, 1)$ is usually called a *po-monoid*.

EXERCISE SM.38$_\square$ Prove that if $b \leq c$ in a po-monoid and b, c are invertible then $c^{-1} \leq b^{-1}$, so that in a po-group G (i.e. a po-monoid that in addition is a group), the inversion map $(-)^{-1} : G^{\mathrm{op}} \to G$ is monotone.

EXERCISE SM.39$_\square$ Let $(G, \leq, \cdot, 1)$ be a po-group; the **positive cone** $G^+ \subseteq G$ is the set $\{x \in G \mid 1 \leq x\}$. Surprisingly, given a group and a positive cone, we can reconstruct the order relation on G: a group equipped with a positive cone is a pair (G, H) where G is a group and $H \subseteq$ is such that

- $1 \in H$;
- if $a, b \in H$, then $ab \in H$;
- if $a \in H$ and $x \in G$, then $x^{-1}ax \in H$;
- if $a, a^{-1} \in G$, then $a = 1$.

Prove that if (G, H) is a group with a positive cone, the relation $a \leq b$ iff $ba^{-1} \in H$ defines a partial order on G that renders $(G, \leq, \cdot, 1)$ a po-group, and H coincides with the positive cone of $(G, \leq, \cdot, 1)$.

EXERCISE SM.40$_\square$ Let P, Q be two posets. Define the **product order** on the Cartesian product $P \times Q$ by $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. Show that the product order $G \times H$ of two po-groups is again a po-group, with the usual group structure on the set $G \times H$.

EXERCISE SM.41$_\square$ Let $(P, \leq)$ be a poset, and let $\sim$ be an equivalence relation on P. One says that $\sim$ is *compatible* with the order relation if $x \leq y$, $x \sim x'$ and $y \sim y'$ imply $x' \leq y'$ or $x' \sim y'$. When this happens the quotient set $P/_\sim$ carries a relation $[x] \leq [y]$ if and only if $x \leq y$ or $x \sim y$; prove that this is a partial order. Prove that the projection to the quotient $P \to P/_\sim$ is a monotone map.

EXERCISE SM.42$_\square$ Let P, Q be two posets. Define the **lexicographic order** on the Cartesian product $P \times Q$ by $(x, y) \leq_{\mathrm{lex}} (x', y')$ if and only if either $x < x'$, or $x = x'$ and $y \leq y'$. Denote $P \times_{\mathrm{lex}} Q$ the set $P \times Q$ equipped with the lexicographic order.

Show that the lexicographic order on $\mathbf{Z} \times \mathbf{Z}$ makes it a po-group. Is it true that more generally, the lexicographic product $\mathbf{Z} \times_{\mathrm{lex}} P$ of two po-groups is a po-group?

EXERCISE SM.43$_\square$ Prove that any group G can be seen as a po-group with the trivial order relation $g \leq h$ if and only if $g = h$; prove that if G is a finite group, the trivial order ($x \leq y$ iff $x = y$) is the only possible po-group structure.

EXERCISE SM.44$_\square$ Let $(P, \leq)$ be a poset, and Q^P the set of monotone mappings $P \to Q$; defines the *standard order* on Q^P by saying that $f \leq g$ when for all $p \in P$ $f(p) \leq g(p)$. Prove that this defines a partial order on Q^P.

We will consider the standard order on Q^P in the particular case $P = Q$ and sometimes restricted to the subset of *invertible* monotone mappings $P \to P$ (i.e. to the set $\text{Aut}(P)$ of 'monotone automorphisms' of P). Prove that P^P is a po-monoid, and $\text{Aut}(P) \subseteq P^P$ is a po-group when both sets are equipped with the standard order.

Definition 3.2. Let $(G, \le)$ be a po-group. A G**-poset**, or a poset *equipped* with a G-action, is a partially ordered set $(P, \le)$ endowed with a po-group homomorphism $a : G \to \text{Aut}(P)$ to the group of order isomorphisms of P with its standard po-group structure.

If a po-group G acts on a poset P, the action of the function $a(g, p)$ is usually denoted as an infix dot $g.p$.

Exercise sm.45□ Show that equivalently, a G-poset is a partially ordered set P together with a group action $G \times P \to P$ which is a monotone map, where on $G \times P$ one puts the product order.

Exercise sm.46□ Prove that the poset $(\mathbf{Z}, \le)$ of integers with their usual order is a $\mathbf{Z}$-poset with the action given by the usual sum of integers. More generally, every po-group is a G-poset, where the action $G \times G \to G$ is the group operation.

Show that the poset $(\mathbf{R}, \le)$ of real numbers with their usual order is a $\mathbf{Z}$-poset for the action given by the sum of real numbers with integers (seen as a subring of real numbers).

Exercise sm.47□ Let G be a po-group acting on a poset P. A G**-fixed point** for a G-poset P is an element $p \in P$ kept fixed by all the elements of G under the G-action. Prove that if P has a top element $\top_P$ or bottom element $\bot_P$, then they both are G-fixed points. Deduce that one can always extend the action of G on P on a larger poset $P_\diamond = P \cup \{+\infty, -\infty\}$ where $-\infty \le x \le +\infty$ for all $x \in P$.

Exercise sm.48□ Given a po-group G and two G-posets P, Q we say that a monotone map $f : P \to Q$ is G-equivariant if for all $g \in G$ one has $f(g.p) = g.f(p)$. More formally, let G be a po-group, and P, Q be two G-posets, respectively with actions $a_P : G \times P \to P$ and $a_Q : G \times Q \to Q$. Then, a monotone map $f : P \to Q$ is G-equivariant if $a_Q(g, fp) = f(a_P(g, p))$

Exercise sm.49□ An equivalence relation $\sim$ on a G-poset P is said to be *compatible* with the G-action if $x \sim y$ implies $g \cdot x \sim g \cdot y$ for any g in G. If $\sim$ is compatible both with the order and with the G-action then the quotient set $P/_\sim$ is naturally a G-poset with the G-action $g \cdot [x] = [g \cdot x]$. Moreover the projection to the quotient is a morphism of G-posets.

Exercise sm.50□ Consider the group $\mathbf{Z}$ of integers as a po-group acting on itself by addition. Prove that there exists a bijection

$$P \cong \{\text{equivariant maps } \varphi : (\mathbf{Z}, \le) \to (P, \le)\} \tag{3.19}$$

arguing as follows: the choice of an element x in a $\mathbf{Z}$-poset P is equivalent to the datum of a $\mathbf{Z}$-equivariant morphism $\varphi_x : (\mathbf{Z}, \le) \to (P, \le)$. (Bonus point if you feel like it: the element x is a $\mathbf{Z}$-fixed point if and only if the corresponding morphism φ factors $\mathbf{Z}$-equivariantly through $(*, \leq)$,)

Exercise sm.51$_\square$ Consider the group $\mathbf{Z}$ of integers as a po-group acting on itself by addition. Prove that a $\mathbf{Z}$-equivariant monotone map $\varphi : (\mathbf{Z}, \leq) \to (P, \leq)$ is either injective or a constant map; as a consequence, if P is a finite poset, there are no nontrivial monotone $\mathbf{Z}$-actions.

Exercise sm.52$_\square$ Show that every equivariant map $\varphi : P \to Q$ where Q is bounded (i.e. it admits a top and a bottom element) extends to a unique $\bar{\varphi} : P_\diamond \to Q$ defining $\bar{\varphi}(\infty) = \top_Q$ and $\bar{\varphi}(-\infty) = \bot_Q$ (see sm.47 for the notation $P_\diamond$).

Definition 3.3. A **quantale** is an algebraic lattice $(Q, \wedge, \vee)$ where *every* subset $S \subseteq Q$ has both an infimum and a supremum. In particular, a quantale Q has a top element $\top_Q$ and a bottom element $\bot_Q$, and it is equipped with a semigroup structure $* : Q \times Q \to Q$ such that both maps $x * \_$ and $\_ * y$ preserve arbitrary joins:

$$x * \Big(\bigvee_{i\in I} y_i\Big) = \bigvee_{i\in I}(x * y_i) \qquad \Big(\bigvee_{i\in I} x_i\Big) * y = \bigvee_{i\in I}(x_i * y). \tag{3.20}$$

The quantale is *unital* when $*$ has an identity element; *commutative* when $*$ is commutative.

Exercise sm.53$_\square$ Show that in a quantale the assignment $(a, b) \mapsto \bigvee\{q \in Q \mid a * q \leq b\}$ defines a binary operation $\_ \multimap \_ : Q \times Q \to Q$ with the property that

$$a * x \leq b \iff x \leq (a \multimap b). \tag{3.21}$$

Similarly, the assignment $(a, b) \mapsto \bigvee\{q \in Q \mid q*a \leq b\}$ defines a binary operation $\_ \multimapinv \_ : Q \times Q \to Q$ with the property that

$$x * a \leq b \iff x \leq (a \multimapinv b). \tag{3.22}$$

Exercise sm.54$_\square$ Show that a two-element set $\{0, 1\}$ has a unique quantale structure if we define $a * b = a \cdot b$ is 1 if and only if both a, b are 1, and 0 otherwise (define $a \Rightarrow b$ in such a way that (3.21) is true, and prove it is the unique possible choice).

Exercise sm.55$_\square$ Let $S = \{0, \epsilon, 1\}$ be a three-element set; prove that the following definitions, packaged in a multiplication table, for $*$ and $\Rightarrow$ equip S with a quantale structure, and show that it is the unique one.

$*$	0	ϵ	1
0	0	0	0
ϵ	0	ϵ	1
1	0	1	1

$*$	0	ϵ	1
0	1	0	0
ϵ	1	ϵ	0
1	1	1	1

A **Heyting algebra** H is a quantale where the $\_ * \_$ operation coincides with the binary meet.

Exercise sm.56$_\square$ Prove that equivalently, a Heyting algebra consists of a meet-semilattice with bottom element, equipped with a binary operation $\_ \multimap \_$ satisfying

$$a \wedge b \leq c \iff c \leq a \multimap b \tag{3.23}$$

Exercise sm.57$_\square$ Prove that a Heyting algebra H admits a *pseudo-complement* operation: for every $a \in H$ there exists a maximum element $\complement a$ with the property that $a \wedge \complement a = 0$.

Exercise sm.58$_\square$ Prove that if H is a Heyting algebra,

- $\complement a$ is uniquely determined by the property that $b \wedge a = 0$ if and only if $b \le \complement a$;
- if $x \le y$, then $\complement y \le \complement x$;
- the assignment $x \mapsto \complement\complement x$ is monotone, and $x \le \complement\complement x$;
- for each $x \in H$, $\complement\complement\complement x = \complement x$;
- for each $a, b \in H$, $\complement\complement(a \wedge b) = \complement\complement a \wedge \complement\complement b$;
- if $a, b \in H$ have pseudo-complements $\complement a$, $\complement b$, then $\complement(a \vee b) = \complement a \wedge \complement b$ and actually more generally the *first de Morgan law* holds:

$$\complement \left(\bigvee_{i\in I} a_i\right) = \bigwedge_{i\in I} \complement a_i \tag{3.24}$$

- for every $a \in H$, $\complement(a \vee \complement a) = 0_H$ (the bottom element of H).

Definition 3.4. A **Boole algebra** is a Heyting algebra that is *complemented*, i.e. the pseudo-complement of a also satisfies the equation $a \vee \complement a = 1$.

Exercise sm.59$_\square$ Prove that a Heyting algebra is a Boole algebra if and only if for every $a \in H$, $\complement\complement a = a$.

Exercise sm.60$_\square$ Prove that in a Boole algebra the *second de Morgan law* holds together with the first:

$$\complement \left(\bigwedge_{i\in I} a_i\right) = \bigvee_{i\in I} \complement a_i \tag{3.25}$$

Exercise sm.61$_\square$ Let $\mathfrak{f}$ be a proper filter in a Boolean algebra B. Prove that the following statements are equivalent.

- $\mathfrak{f}$ is maximal;
- $\mathfrak{f}$ is prime;
- for every $a \in B$ either a or $\complement a \in \mathfrak{f}$.

(This generalises or.62).

Every Heyting algebra H contains a maximal Boole algebra:

Exercise sm.62$_\square$ Prove that $BH = \{a \in H \mid \complement\complement a = a\} \subseteq H$ is a Boole algebra.

Exercise sm.63$_\square$ Prove that the following conditions on a Heyting algebra L are equivalent:

- the second de Morgan law (3.25) holds;
- for each $a \in H$, $\complement\complement a \vee \complement a = 1_H$;
- every element of BH, as defined above, has a complement in H;
- the identity $\complement\complement(a \vee b) = \complement\complement a \vee \complement\complement b$ holds for all $a, b \in H$;
- BH is a sublattice of H.

Exercise sm.64$_\square$ Show that any po-group G (written multiplicatively) admitting all sups and inf is a quantale if we define $a * b$ as the group multiplication ab, and $a \multimap b = a^{-1}$, $a \multimapinv b = ba^{-1}$ (cf. (3.21) and (3.22)).

Exercise sm.65$_\square$ Let Q be a quantale, $a, b, c \in Q$. Prove the following facts:

- $a * (a \multimap b) \leq b$;
- $(a \multimapinv b) * a \leq b$;
- $b \multimap (\multimap c) = (a * b) \multimap c$;
- $a \multimapinv (b \multimapinv c) = (a * b) \multimapinv c$;
- $a \multimap (b \multimapinv c) = b \multimapinv (a \multimap c)$;
- $a * (a \multimap b) = b$ if and only if there exists c such that $a * c = b$;
- $(a \multimapinv b) * a = b$ if and only if there exists c with $c * a = b$.

Exercise sm.66$_\square$ Let M be a monoid, and consider the set PM of subsets of M. Define a binary operation

$$A * B := \{ab \mid a \in A, b \in B\}, \tag{3.26}$$

and show that $(PM, *)$ is a quantale when considered with the natural order given by the inclusion of subsets.

Exercise sm.67$_\square$ (Pataraia's constructive proof of Knaster-Tarski theorem (cf. po.14)). Recall the definition of *po-monoid* from Definition 3.1; a po-monoid $(M, \cdot, 1)$ is called **directed complete** when every directed subset admits a supremum, and it is said to have a **zero** when there exists an element $z \in M$ such that $xz = zx = z$ for all $\in M$. (For example, the po-monoid of natural numbers $(\mathbf{N}, \cdot, 1)$ *including zero* is directed complete and has a zero.)

- Show that if M is a directed complete monoid such that $1 \leq x$ for all $x \in M$, then M has a zero. [Hint: show that M is a directed set, and let $z = \bigvee M$.]
- Use the above point to prove that any monotone map $f : D \to D$ defined on a dcpo D with a bottom element, admits a minimal fix point. [Hint: let $S = \{x \in D \mid x \leq fx\}$ be the set of preix points of f; let PS be the set of monotone endofunctions h such that id $\leq h$ and $h(s) = s$ for all $s \in S$. Apply the previous point when $M = PS$.]

CHAPTER 4

Linear Algebra, done hard

Recall the definition of **ring**, **ring homomorphism**, **ideal**, **field** and get back at whatever amount of Linear Algebra you have been previously exposed, better if from an Abstract Algebra book. You're going to need it.

Exercise LA.1 $\square$ Let R be a unital ring; prove that there exists a unique ring homomorphism $\eta_R : \mathbf{Z} \to R$, called the **characteristic** homomorphism.

Find the kernel of η_R when $R = \mathbf{Z}, \mathbf{Q}$ and when $R = \mathbf{Z}/n\mathbf{Z}$ is the ring of integers modulo n. Prove that $\ker \eta_R$ is an ideal of $\mathbf{Z}$ generated by a single element $q \in \mathbf{Z}$.

Exercise LA.2 $\square$ Given a unital ring R, its characteristic is the minimal generator q of $\ker \eta_R$.

Prove that a field has characteristic either zero or a prime number.

A **ring extension** of a commutative ring R is a commutative ring E of which R is a subring. In other words, a ring extension is an injective ring homomorphism $R \to E$.

Exercise LA.3 $\square$ Let F be a field; prove that a ring homomorphism $F \to E$ is either the constant at zero homomorphism, or a ring extension (in particular, if we insist on a ring homomorphism to preserve the multiplicative unit, all ring homomorphisms from a field are ring extensions).

We denote a field extension E of a field F as $E|F$.

Exercise LA.4 $\square$ Prove that an extension $E|F$ makes E a vector space over F: the extension is called **finite** if E has finite dimension over F; note that being finite for E depends on F, by finding a field F_1 such that $E|F_1$ is finite, and an extension $F_1|F_2$ which is not finite, so that $E|F_2$ is not finite.

Exercise LA.5 $\square$ Prove that there exists a polynomial p wih real coefficients, of degree at most 3, such that $p(x+1) - p(x) = x^2$. Determine $p(x)$. Use this result to compute the sum $\sum_{k=1}^{n} k^2$ of the squares of the first n positive integers.

Exercise LA.6 $\square$ Consider the vector space $L = \mathbf{R}[X]_{<n}$ whose elements are polynomials of degree at most $n-1$; prove that

- the standard basis: i.e. the set $\{1, x, \dots, x^{n-1}\}$ is a basis of L; the coordinates of a vector in this basis are its coefficients.
- the Taylor basis: for $a \in \mathbf{R}$, the set $\{1, x-a, (x-a)^2, \dots, (x-a)^{n-1}\}$ is a basis of L; the coordinates of a vector $f \in L$ in this basis ar given by its subsequent derivatives evaluated in a: $\{f(a), f'(a), \frac{f''(a)}{2}, \dots, \frac{f^{(n-1)}(a)}{(n-1)!}\}$.

Is this still true if instead of polynomials with real coefficients, we take polynomials wih coefficients in a finite field?

- interpolation basis: if $a_1, \dots, a_n \in \mathbf{R}$ are pairwise distinct elements of L, define
$$g_i(x) := \prod_{j \neq i} \frac{x - a_j}{a_i - a_j}. \tag{4.1}$$
Then $\{g_1, \dots, g_n\}$ is a basis of L; the coordinates of a vector $f \in L$ in this basis are given by the values $\{f(a_1), \dots, f_{a_n}\}$.

Exercise la.7$_\square$ Let $W \leq V$ be a subspace inclusion; the **quotient** vector space V/W consists of the vector space of equivalence classes of vectors of V by the relation
$$v \sim_W v' \iff v - v' \in W. \tag{4.2}$$
Prove that $\sim_W$ is an equivalence relation, and in fact a congruence ($v \sim_W v'$ implies $u+v \sim_W u+v'$ for all $u \in V$); prove that the set of $\sim_W$-equivalence classes becomes a vector space if we define the sum $[v]+[v']$ as $[v+v']$, and the scalar multiplication $a[v]$ as $[av]$.

Exercise la.8$_\square$ Find a geometric interpretation for the following quotient vector spaces:

- $\mathbf{R}^2/\langle(0, 1)\rangle$;
- $\mathbf{R}^3/\langle(1, 0, 0), (0, 1, 1)\rangle$.

(Hint: represent the elements of the quotient space V/W, i.e. the $\sim_W$-equivalence classes, as $[v] = v + W = \{v + w \mid w \in W\}$, i.e. as subsets of V resulting as translations of W by vectors of v; observe that in no case $v + W$ is a vector subspace of V: why? Show that for each $v \in V$, $[v]$ equals $[v_\perp]$, where $v_\perp$ is a vector perpendicular to all vectors of W, in the sense that the scalar product $v \cdot w$ is zero for every $w \in W$; deduce that V/W can be identified with the set of such perpendicular vectors.)

Exercise la.9$_\square$ Let $f : V \to W$ be a linear maps between F-vector spaces; the **cokernel** of f is the quotient space $W/\text{im}\, f$, where two vectors $w, w' \in W$ are identified if and only if their difference lies in the image of f;

- prove that f is surjective if and only if coker f is the zero vector space;
- what is the cokernel of the zero map $0 : V \to W$? What is the cokernel of the inclusion of $\langle v\rangle$ in $\mathbf{R}^3$ as v varies through the vectors of $\mathbf{R}^3$?

Definition 4.1. Let V be a F-vector space. Define the vector space V^* (called the **dual** of V) as the set $\hom(V, F)$ of all F-linear homomorphisms from V to F, equipped with the obvious vector space structure given by $(\alpha+\beta)(v) = \alpha(v)+\beta(v)$ and $(t \cdot \alpha)(v) = t \cdot \alpha(v)$ for every $t \in F$, $v \in V$, $\alpha, \beta : V \to F$.

Exercise la.10$_\square$ Show that this defines a structure of F-vector space on $\hom(V, F)$.

Exercise la.11$_\square$ Let V be a finite-dimensional F-vector space. Let $\mathcal{V} = \{v_1, \dots, v_n\}$ be a basis of V, and let $\mathcal{V}^* = \{\alpha_1, \dots, \alpha_n\}$ be the set of vectors in V^*

defined as follows: $\alpha_i(v_j) = 1$ if $i = j$ and 0 otherwise. Show that $\mathcal{V}^*$ is a basis of V^*, called the **dual basis** associated to $\mathcal{V}$.

Exercise LA.12$_\square$ Show that to each F-linear map $f : V \to W$ of vector spaces one can associate the **transposed** map $f^* : W^* \to V^*$ sending an element $\alpha \in W^*$ to the linear map

$$\alpha \circ f : V \xrightarrow{f} W \xrightarrow{\alpha} F \tag{4.3}$$

Exercise LA.13$_\square$ Show that there exists a function

$$V^* \times V \xrightarrow{\circ} F \tag{4.4}$$

called **canonical duality** which enjoys the following properties:

- bilinearity:
- nondegenerate:

Let S be a subset of a vector space V; we define the **orthogonal** of S in V^* as the subspace of V^* as

$$S^\perp := \{\alpha \in V^* \mid \alpha \circ v = 0, \forall v \in S\} \tag{4.5}$$

Similarly, for a subset T of V^* we define the subspace $T^\perp$ of V as

$$T^\perp := \{v \in V \mid \alpha \circ v = 0, \forall \alpha \in T\} \tag{4.6}$$

Exercise LA.14$_\square$ Prove that

- if $A \subseteq B$ then $B^\perp \subseteq A^\perp$;
- for every $S \subseteq V$, $S^{\perp\perp} = \langle S \rangle$ is the subspace generated by S (similarly for $T \subseteq V^*$); moreover, $S^{\perp\perp\perp} = S^\perp$ (and similarly for $T \subseteq V^*$).
- $(A \cap B)^\perp = A^\perp + B^\perp$ and $(A + B)^\perp = A^\perp \cap B^\perp$ for subspaces $A, B \leq V$.

In particular passing to the orthogonal induces an antiisoomrphism between the lattices of subspaces of V and V^*.

Exercise LA.15$_\square$ Let $f : V \to W$ be a linear map; show that $\ker(f^*) = (\operatorname{im} f)^\perp$ and $\operatorname{im}(f^*) = (\ker f)^\perp$. Deduce that the rank of f equals the rank of f^*

Exercise LA.16$_\square$ Show that given a subspace $A \leq V$ of a finite dimensional k-vector space, the inclusion $A \hookrightarrow V$ induces a surjective linear map $V^* \to A^*$, having kernel $A^\perp$; deduce that $(V/A)^* \cong A^\perp$ as vector spaces.

Exercise LA.17$_\square$ More generally, given subspaces $U \leq W$ of the same V show that $(W/U)^*$ is isomorphic to $U^\perp/W^\perp$.

With this –but also directly– show that

- $((U + W)/W) \cong W^\perp/(U^\perp \cap W^\perp)$;
- $(W/(U \cap W))^* \cong (U^\perp + W^\perp)/W^\perp$.

Exercise LA.18$_\square$ Show that the vector space $\mathbf{Q}[X]$ of polynomials with rational coefficients is not isomorphic to its dual $\mathbf{Q}[X]^*$.

Exercise LA.19$_\square$ Sia $g : V \to W$ una mappa lineare; dimostrare che l'equazione $g(v) = w$ ammette una soluzione, fissao $w \in W$, se e solo se $w \in \ker g^*$, dove $g^* : W^* \to V^*$ è la trasposta di g.

Exercise LA.20$_\square$ (Product of vector spaces.) Let I be a set and V_i a family of F-vector spaces indexed by I; the **product** $\prod_{i\in I} V_i$ of the family V_i is the set obtained from the cartesian product of the V_i and equipped with componentwise sum and scalar multiplication.

Show that the canonical projections $\pi_j : \prod_i V_i \to V_j$ for $j \in I$ is F-linear.

Show that $\prod_i V_i$ enjoys the following property:

> Given any other family of F-linear maps $f_i : W \to V_i$, there exists a unique $f : W \to \prod_i V_i$ which is F-linear and makes the following diagram commute for every $j \in I$:
>
> $$\begin{tikzcd} W \arrow[rr, dotted, "f"] \arrow[dr, "f_j"'] & & \prod_i V_i \arrow[dl, "\pi_j"] \\ & V_j & \end{tikzcd} \tag{4.7}$$

Usually the notation for such an f is $\prod_i f_i$

Exercise LA.21$_\square$ (Coproduct of vector spaces.) Let I be a set and V_i a family of F-vector spaces indexed by I; the **coproduct** (or *direct sum*) $\sum_{i\in I} V_i$ of the family V_i is the subset of $\prod_i V_i$, equipped with componentwise sum and scalar multiplication, whose elements are those sequences of vectors $(v_i \mid i \in I, v_i \in V_i)$ such that $v_i \neq 0$ for at most a finite number of indices.

Show that this is indeed a vector subspace of $\prod_i V_i$, and that the inclusions $\iota_j : \sum_i V_i \to V_j$ are F-linear for every $j \in I$.

Show that $\sum_i V_i$ enjoys the following property:

> Given any other family of F-linear maps $f_i : V_i \to W$, there exists a unique $f : \sum_i V_i \to W$ which is F-linear and makes the following diagram commute for every $j \in I$:
>
> $$\begin{tikzcd} & V_j \arrow[dl, "\iota_j"'] \arrow[dr, "f_j"] & \\ \sum_i V_i \arrow[rr, "f"'] & & W \end{tikzcd} \tag{4.8}$$

Usually the notation for such an f is $\sum_i f_i$.

Exercise LA.22$_\square$ Prove that if I is a finite set and V_i a family of vector spaces indexed by I, then there is an isomorphism $\prod_i V_i \cong \sum_i V_i$ (hint: start from $I = \{1, 2\}$ a set with two elements).

Exercise LA.23$_\square$ Let S be a finite set of cardinality n; consider the vector space $V = \{f : 2^S \to \mathbf{R}\}$ of all functions from 2^S to $\mathbf{R}$, and the linear map $\varphi : V \to V$ given by

$$\varphi(f) : T \mapsto \sum_{Y \supseteq T} fY \tag{4.9}$$

Show that φ is an isomorphism of vector spaces. (Hint: induction on n.)

Definition 4.2. Let $k \geq 1$ be an integer, and $\mathcal{W} = (W_1, \dots, W_k)$ and $\mathcal{W}' = (W_1', \dots, W_k')$ be two k-tuples of subspaces of the same vector space V; we say that $\mathcal{W}, \mathcal{W}'$ are **concordant** if there exists an invertible linear map $\varphi : V \to V$ such that $\varphi(W_i) = W_i'$ for each $1 \leq i \leq k$.

Exercise la.24$_\square$ Show that two tuples of subspaces $\mathcal{W}, \mathcal{W}'$ as above are concordant if and only if for each $1 \leq k \leq n$ and each choice of indices $(i_1, \dots, i_k)$ there is a linear isomorphism $\varphi_{i_1 \dots i_k} : W_{i_1, \dots, i_k} \to W'_{i_1, \dots, i_k}$, where we denote $W_{i_1 \dots i_k} = W_{i_1} \cap \dots \cap W_{i_k}$.

Exercise la.25$_\square$ Show that a k-tuple $\mathcal{W} = (W_1, \dots, W_n)$ of subspaces of V is equivalently described by its **nerve**: let again $W_{i_1 \dots i_r} := W_{i_1} \cap \dots \cap W_{i_r}$ and consider the diagram

$$\begin{tikzcd}
\sum_{i=1}^{k} W_i \arrow[r, shift left=3, "\pi_1"] \arrow[r, leftarrow, "\iota_1" description] \arrow[r, shift right=3, "\pi_2"'] & \sum_{i_1<i_2} W_{i_1 i_2} \arrow[r, shift left=6, "\pi_1"] \arrow[r, leftarrow, shift left=3, "\iota_1" description] \arrow[r, "\pi_2" description] \arrow[r, leftarrow, shift right=3, "\iota_2" description] \arrow[r, shift right=6, "\pi_3"'] & \sum_{i_1<i_2<i_3} W_{i_1 i_2 i_3} \arrow[r, shift left=6, "\pi_1"] \arrow[r, leftarrow, shift left=3, "\iota_3" description] \arrow[r, phantom, "\cdots"] \arrow[r, leftarrow, shift right=3, "\iota_1" description] \arrow[r, shift right=6, "\pi_4"'] & \cdots
\end{tikzcd} \tag{4.10}$$

where the maps pointing to the right are induced by projections from the intersection of i elements of the tuple to the intersection of $(i+1)$ elements of the tuple, and the arrows pointing to the left are induced by the inclusions from an intersection of i elements to an intersection of $(i-1)$ elements of the tuple.

Given two of such diagrams, filled by the dotted arrows $\varphi_{i_1, \dots i_r}$ below,

$$\begin{tikzcd}
\sum_{i=1}^{k} W_i \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d, dotted, "\sum_i \varphi_i"'] & \sum_{i_1<i_2} W_{i_1 i_2} \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] \arrow[d, dotted, "\sum \varphi_{i_1 i_2}"'] & \sum_{i_1<i_2<i_3} W_{i_1 i_2 i_3} \arrow[r, shift left=6] \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[r, shift right=6] \arrow[d, dotted, "\sum \varphi_{i_1 i_2 i_3}"'] & \cdots \\
\sum_{i=1}^{k} W_i' \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \sum_{i_1<i_2} W'_{i_1 i_2} \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & \sum_{i_1<i_2<i_3} W'_{i_1 i_2 i_3} \arrow[r, shift left=6] \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[r, shift right=6] & \cdots
\end{tikzcd} \tag{4.11}$$

the two k-tuples that they represent are concordant if and only if for each choice of homonymous horizontal arrows pointing in the same direction, the resulting diagram is commutative: in other words, all diagrams

$$\begin{tikzcd}
\sum_{i=1}^{k} W_i \arrow[r, "\pi_1"] \arrow[d, "\sum_i \varphi_i"'] & \sum_{i_1<i_2} W_{i_1 i_2} \arrow[d, "\sum \varphi_{i_1 i_2}"'] \\
\sum_{i=1}^{k} W_i' \arrow[r, "\pi_1'"'] & \sum_{i_1<i_2} W'_{i_1 i_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum_{i=1}^{k} W_i \arrow[r, "\pi_2"] \arrow[d, "\sum_i \varphi_i"'] & \sum_{i_1<i_2} W_{i_1 i_2} \arrow[d, "\sum \varphi_{i_1 i_2}"'] \\
\sum_{i=1}^{k} W_i' \arrow[r, "\pi_2'"'] & \sum_{i_1<i_2} W'_{i_1 i_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum_{i=1}^{k} W_i \arrow[d, "\sum_i \varphi_i"'] & \sum_{i_1<i_2} W_{i_1 i_2} \arrow[l, "\iota_1"'] \arrow[d, "\sum \varphi_{i_1 i_2}"'] \\
\sum_{i=1}^{k} W_i' & \sum_{i_1<i_2} W'_{i_1 i_2} \arrow[l, "\iota_1"]
\end{tikzcd}$$

etc., are commutative.

Exercise la.26$_\square$ Let q be a prime number, and $n \geq 1$ an integer; build a field having exactly q^n elements by considering the roots of the polynomial $X^{q^n} - X$; prove that every finite field extension $\mathbf{Z}/q\mathbf{Z}$ arises in this way, or in other words, prove that if F is a finite field of characteristic q, then it has q^n elements for some $n \geq 1$, and it is *the unique* such field up to isomorphism.

Exercise LA.27$_\square$ Let $n \geq 1$ be an integer. Prove that the number of subspaces of the vector space $V = (\mathbf{Z}/q\mathbf{Z})^n$ having dimension $k \leq n$ is the q-**binomial coefficient**

$$\binom{n}{k}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q} \tag{4.12}$$

where for every real number q (so a fortiori for an integer) the quantity $[r]!_q$ is defined as $(1+q)(1+q+q^2)\dots(1+q+\dots+q^{r-1})$;

Exercise LA.28$_\square$ Prove that the set of all invertible linear maps of $V = (\mathbf{Z}/q\mathbf{Z})^n$ onto itself has $q^{\binom{n}{2}}(q-1)^n[n]!_q$ elements.

Exercise LA.29$_\square$ Let F be a field, and G the subgroup of $n\times n$ matrices having the property that

> There exists *exactly* one 1 in each row and in each column.

Prove that there exists a group isomorphism between G and the symmetric group $S(n)$ of all permutations of an n-element set. Is the subgroup G normal in $GL_n(F)$? Are the matrices in G diagonalizable? Who is the characteristic polynomial of a matrix $\Sigma \in G$?

Exercise LA.30$_\square$ Let F be a field; interpret the determinant

$$\det := \sum_{\sigma\in S(n)} (-1)^{|\sigma|} \prod_{i=1}^n a_{i,\sigma i} \tag{4.13}$$

of a generic $n \times n$ matrix with coefficients in F as a polynomial in the n^2 indeterminates $\{a_{ij} \mid 1 \leq i, j \leq n\}$.

Prove that det is an irreducible polynomial in $F[a_{ij} \mid 1 \leq i, j \leq n]$. (Hint: induction on n, starting from $n = 2$.)

Exercise LA.31$_\square$ Consider the F-vector space $F^{V\times W} = \sum_{V\times W} F$, and the subspace generated by the relations

$$\begin{cases} (v_1, w) + (v_2, w) \sim (v_1 + v_2, w), \\ (v, w_1) + (v, w_2) \sim (v, w_1 + w_2), \\ c(v, w) \sim (cv, w), \\ c(v, w) \sim (v, cw). \end{cases} \tag{4.14}$$

in $F^{V\times W}$. Show that this quotient satisifies the universal property of the tensor product $V \otimes W$ of V, W:

> To each *bilinear* map $\varphi : V \times W \to U$ corresponds a unique *linear* map $\overline{\varphi} : V \otimes W \to U$, bijectively.

Prove that when V, W have finite dimension over the field F, there is an isomorphism $V \otimes W \cong \mathrm{Bil}(V \times W, F)^*$ (on the right hand side: the dual of the vector space of bilinear maps $V \times W \to F$).

Exercise LA.32$_\square$ Prove that $V \otimes W$ has a basis $\mathcal{V} \otimes \mathcal{W} = \{v_i \otimes w_j\}$, when v_i runs over a basis $\mathcal{V}$ of V, and w_j runs over a basis $\mathcal{W}$ of W. In the identification $V \otimes W \cong \operatorname{Bil}(V \times W, F)^*$, to which elements of $\operatorname{Bil}(V \times W, F)^*$ does $v_i \otimes w_j$ correspond?

Prove that $F \otimes F \cong F$; prove that $V \otimes F \cong V$ for each vector space V; prove that $V \otimes (W \otimes Z) \cong (V \otimes W) \otimes Z$; prove that $V \otimes W \cong W \otimes V$.

Exercise LA.33$_\square$ Prove that each pair of linear maps $f : V \to V'$, $g : W \to W'$ induces a linear map $f \otimes g : V \otimes W \to V' \otimes W'$; how is this map defined on the basis $\mathcal{V} \otimes \mathcal{W}$ of $V \otimes W$? Assume V, V', W, W' all have finite dimension and fix bases $\mathcal{V}, \mathcal{W}, \mathcal{V}', \mathcal{W}'$ for all vector spaces in question; let A be the matrix of f, and B be the matrix of g, an find an expression for the matrix of $f \otimes g$ in terms of A, B; the matrix $A \otimes B$ is called the **Kronecker product** of A, B.

Exercise LA.34$_\square$ Show the following properties of the Kronecker product:

- given matrices A, B, C, prove that if A, B have the same size,
$$(A + B) \otimes C = A \otimes C + B \otimes C \quad C \otimes (A + B) = C \otimes A + C \otimes B;$$
- given matrices A, B, C prove that
$$A \otimes (B \otimes C) = (A \otimes B) \otimes C \tag{4.15}$$
- if A, B, C, D are matrices such that the matrix products AC, BD exist, then $(A \otimes B)(C \otimes D) = AC \otimes BD$; as a corollary, if A, B are both invertible, so is $A \otimes B$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
- if A, B are matrices, respecively $n \times n$ and $m \times m$, then

$$\det(A \otimes B) = (\det A)^m (\det B)^n \quad \operatorname{trace}(A \otimes B) = \operatorname{trace}(A)\operatorname{trace}(B). \tag{4.16}$$

Exercise LA.35$_\square$ The **tensor algebra** over a vector space V is defined as the vector space
$$TV = \sum_{n \geq 0} V^{\otimes n} \tag{4.17}$$
equipped with the *tensor product* operation: two elements $x_1 \otimes \cdots \otimes x_n$ and $y_1 \otimes \cdots \otimes y_m$ can be multiplied into an element
$$x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m \in V^{\otimes(n+m)} \tag{4.18}$$
Prove that the tensor product operation equips TV with a **graded algebra** structure.

Prove that when V is finite dimensional, say with dimension d, TV is isomorphic –as algebra– to the algebra of **noncommutative polynomials** $k\{X_1, \ldots, X_d\}$.

Exercise LA.36$_\square$ For what has been shown in LA.35 TV is a ring;

- Describe the ideal I in $(TV, \otimes)$ generated by the set $S = \{v \otimes w - w \otimes v \mid v, w \in V\}$; prove that the quotient TV/I is isomorphic to the ring o polynomials $k[X_1, \ldots, X_d]$; define explicitly the isomorphism.
- Describe the ideal J generated by the set $\{v \otimes w + w \otimes v \mid v, w \in V\}$; prove that J is also generated by the set $\{v \otimes v \mid v \in V\}$;
- Fix a basis $\{e_1, \ldots, e_n\}$ of V. Describe the ideal L generated by the set $\{e_i \otimes e_i - 1, e_i \otimes e_j + e_j \otimes e_i \mid e_i, e_j \in V\}$.

The quotient TV/J is the **exterior** (or *Grassmann*) *algebra* $\bigwedge(V)$ of V constructed from the tensor algebra. The quotient TV/L is the **Clifford algebra** $Cl(V)$ of V. See LA.37 for more information on the Grassmann algebra; see Definition 4.5 for more information on the Clifford algebra.

Exercise LA.37□ Prove that the tensor product operation on TV induces a multiplication

$$\_\wedge\_ : \ \bigwedge(V)\times\bigwedge(V)\longrightarrow\bigwedge(V) \tag{4.19}$$

on the quotient that defines $\bigwedge(V)$.

Exercise LA.38□ Prove that in fact the ideal J such that $\bigwedge(V)\cong TV/J$ is a *graded* ideal: there exists a decomposition $J=\bigoplus_{r\geq 0}J_r$ where $J_r := J\cap V^{\otimes r}$ such that if we pose $\bigwedge_r(V) := V^{\otimes r}/J_r$ there exists a decomposition

$$\bigwedge(V)\cong\bigoplus_{r\geq 0}\bigwedge_r(V) \tag{4.20}$$

Note that $\bigwedge_0(V)\cong F$ (the base field) and $\bigwedge_1(V)\cong V$, so that there exists a canonical F-linear map $j: V\hookrightarrow\bigwedge(V)$.

Exercise LA.39□ Prove that if V has dimension d and $r>d$, then $\bigwedge_r(V)\cong(0)$ (the zero vector space). In fact, prove that a basis of $\bigwedge_r(V)$ can be found, once a basis $\mathcal{V}=\{e_1,\dots,e_d\}$ of V has been fixed, taking the elements

$$e_{i_1}\wedge\cdots\wedge e_{i_r} \tag{4.21}$$

where $1\leq i_1<\cdots<i_r\leq d$. From this, $\dim_F\bigwedge_r(V)=\binom{d}{r}$.

Exercise LA.40□ Prove the universal property of $\bigwedge(V)$:

> Gven a F-algebra $(A,\cdot)$ and a F-linear map $f: V\to A$ such that for each $v\in V$, $fv\cdot fv=0$ in A, there exists a unique extension $\bar{f}:\bigwedge(V)\to A$ in
>
> $$\begin{tikzcd} & V \arrow[dl, "j"'] \arrow[dr, "f"] & \\ \bigwedge(V) \arrow[rr, dotted, "\bar{f}"'] & & A \end{tikzcd} \tag{4.22}$$
>
> that makes the diagram commute.

Exercise LA.41□ As a corollary of the above universal property, given any linear map $f: V\to W$ of vector spaces, there exists a unique $\overline{jf}=\bigwedge f:\bigwedge(V)\to\bigwedge(W)$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "j_V"'] & W \arrow[d, "j_W"] \\ \bigwedge(V) \arrow[r, "\bigwedge f"'] & \bigwedge(W) \end{tikzcd} \tag{4.23}$$

is commutative.

Exercise LA.42□ Show that

- $\bigwedge f$ is uniquely determined by the request that $\bigwedge f(v_1 \wedge \cdots \wedge v_r) = fv_1 \wedge \cdots \wedge fv_r$ for every $r \geq 1$ and $v_1, \dots, v_r \in V$;
- $\bigwedge(g \circ f) = \bigwedge g \circ \bigwedge f$ for;
- $\bigwedge(\mathrm{id}_V) = \mathrm{id}_{\bigwedge V}$.

Exercise la.43$_\square$ Let $x \in \bigwedge_r(V), y \in \bigwedge_s(V)$; then $y \wedge x = (-1)^{rs} x \wedge y$; in particular, the exterior product is *anticommutative*, i.e. for any vectors v, w one has $v \wedge w = -w \wedge v$.

Exercise la.44$_\square$ Let $\{W_i \mid i \in I\}$be a family of F-vector spaces; prove the isomorphism

$$V \otimes \big(\textstyle\bigoplus_{i \in I} W_i \big) \cong \textstyle\bigoplus_{i \in I} V \otimes W_i \tag{4.24}$$

and deduce that

$$\textstyle\bigwedge(V) \otimes \bigwedge(W) \cong \bigoplus_{r,s \geq 0} \bigwedge_r(V) \otimes \bigwedge_s(W) \tag{4.25}$$

a relation from which one can prove the **exponential property** for $\bigwedge(\_)$:

$$\textstyle\bigwedge(V \oplus W) \cong \bigwedge(V) \otimes \bigwedge(W) \tag{4.26}$$

Use the exponential property to find the dimension of $\bigwedge(V)$ by induction on $d = \dim_F V$.

Exercise la.45$_\square$ Let $f : V \to W$ be a linear map of F-vector spaces; consider the induced map

$$\textstyle\bigwedge f : \ \bigwedge(V) \longrightarrow \bigwedge(W) \tag{4.27}$$

between the exterior algebras.

Find a matrix expression for $\bigwedge f$ once bases $\mathcal{V} = \{v_1, \dots, v_n\}$ and $\mathcal{W} = \{w_1, \dots, w_m\}$ of V, W respecively have been fixed.

Definition 4.3. A **super vector space** is a vector space V over a field k that can be decomposed as a direct sum $V = V_0 \oplus V_1$; the elements of the form $(v, 0)$ for $v \in V_0$ are called *even*, while the elements $(0, v)$ for $v \in V_1$ are called *odd*. A homomorphism of super vector spaces V, W is a linear map $f : V \to W$ that preserves the parity of elements (equivalently: such that $f(V_0) \subseteq W_0, f(V_1) \subseteq W_1$).

Exercise la.46$_\square$ Define the **dimension** of a super vector space V to be $\dim V = \dim V_0 - \dim V_1$. Define the direct sum and intersection of super vector subspaces (define them first!), or prove that it doesn't always exist. Does the Grassmann formula still hold true?

Exercise la.47$_\square$ How to define the cartesian product of super vector spaces V, W? How to define their direct sum? Is it still true that finite products and finite coproducts coincide, $\prod_{i=0}^n V_i \cong \sum_{i=0}^n V_i$, as it happens for vector spaces? [Hint: the $\prod$ and $\sum$ constructions should coincide with the usual ones for 'purely even' super vector spaces, i.e. it should be true that

$$(V, 0) \times (W, 0) \cong (V \times W, 0) \qquad (V, 0) \oplus (W, 0) \cong (V \oplus W, 0). \tag{4.28}$$

Given this, find a definition for the 'purely odd' case and for the mixed case.]

Definition 4.4. An **exact sequence** of linear maps is a string of vector spaces and composable linear maps

$$\mathfrak{v} : V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} V_{n+1} \tag{4.29}$$

such that $\ker f_{i+1} = \operatorname{im} f_i$ for every $i = 0, \dots, n$.

Note that this defining property implies in particular that $f_{i+1} \circ f_i = 0$; when this weaker condition is satisfied we say that the sequence of spaces and linear maps forms a **chain complex** (so every exact sequence is a chain complex, but the converse does not hold). If $\mathfrak{v}$ is a complex, the quotient spaces $H^i(\mathfrak{v}) = \ker f_{i+1}/\operatorname{im} f_i$ are an intrinsic invariant of the complex and of great theoretical interest.

Exercise la.48□ Let

$$0 \to U \to V \to W \to 0 \tag{4.30}$$

be an exact sequence; show that $\dim U - \dim V + \dim W = 0$; more generally, let

$$\mathfrak{v} : 0 \to V_1 \to V_2 \to \cdots \to V_n \to 0 \tag{4.31}$$

be an exact sequence; show that the alternating sum of dimensions $\sum(-1)^i \dim V_i$ equals 0.

This leads to the following definition, valid for every complex: let

$$\mathfrak{v} : 0 \to V_1 \to V_2 \to \cdots \to V_n \to 0 \tag{4.32}$$

be a complex of vector spaces; we define the following objects:

- the i-th **cohomology group** $H^i(\mathfrak{v})$ is defined as the quotient $\ker f_{i+1}/\operatorname{im} f_i$, and the total cohomology of $\mathfrak{v}$ is defined as $\sum H^i(\mathfrak{v})$;
- the i-th *Betti number* $b_j(\mathfrak{v})$ of $\mathfrak{v}$ is defined as $\dim H^i(\mathfrak{v})$;
- the **Euler characteristic** of $\mathfrak{v}$ is defined as the signed sum $\sum(-1)^j b_j(\mathfrak{v})$.

Exercise la.49□ Now, let

$$\mathfrak{v} : 0 \to V_1 \to V_2 \to \cdots \to V_n \to 0 \tag{4.33}$$

be a complex and let $d_j = \dim V_j$; show that the following conditions are equivalent:

- the complex $\mathfrak{v}$ is exact;
- each $H^i(\mathfrak{v})$ is zero;
- the Euler characteristic of th complex, $\sum(-1)^j d_j$, is zero.

Let R be a (commutative, unital) ring; an R-**module** consists of the exact same thing a vector space is, with the only difference that the 'scalar multiplication' operation now takes values in R, a ring that is not necessarily a field. This seemingly innocuous difference generates a lot of differences between the theory of modules and the theory of vector spaces.

Exercise la.50□ What is a module over the ring $\mathbf{R}[X]$ of polynomials with real coefficients?

Exercise la.51□ Let I be a set; a *family* of R-modules M_i is a set $\{M_i \mid i \in I\}$ of modules over R; the **direct sum** of a family of R-modules is a module $\sum_{i\in I} M_i$ such that the following properties are satisfied:

- There exists a family of R-linear maps $\{\iota_j : M_j \to \sum_{i\in I} M_i \mid j \in I\}$ such that
- given any other family of R-linear maps $\{f_j : M_j \to X \mid j \in I\}$, there exists a unique $\bar{f} : \sum_{i\in I} M_i \to X$ such that $\bar{f} \circ \iota_j = f_j$.

Prove that whenever another object $\sum^{\star}_{i\in I} M_i$ satisfies the same properties, then there exists a unique isomorphism $\sum^{\star}_{i\in I} M_i \cong \sum_{i\in I} M_i$.

Exercise la.52$_\square$ The **direct product** of a family of R-modules M_i is the module $\prod_{i\in I} M_i$ such that

- There exists a family of R-linear maps $\{\pi_j : \prod_{i\in I} M_i \to M_j \mid j \in I\}$ such that
- given any other family of R-linear maps $\{f_j : X \to M_j \mid j \in I\}$, there exists a unique $\bar{f} : X \to \prod_{i\in I} M_i$ such that $\pi_j \circ \bar{f} = f_j$.

Prove that whenever another object $\prod^{\star}_{i\in I} M_i$ satisfies the same properties, then there exists a unique isomorphism $\prod^{\star}_{i\in I} M_i \cong \prod_{i\in I} M_i$.

Exercise la.53$_\square$ Prove that when I is a finite set, $\prod_{i\in I} A_i \cong \sum_{i\in I} A_i$.

Exercise la.54$_\square$ Is it true or false that $\prod_{i\in I} \sum_{j\in J} A_{ij} \cong \sum_{j\in J} \prod_{i\in I} A_{ij}$, for every family fof R-modules $\{A_{ij} \mid (i,j) \in I \times J\}$?

Exercise la.55$_\square$ Prove that given an R-module M and a set I, one has $M \cong \sum_{i\in I} M_i$ if and only if one can find homomorphisms $\mu^j : M_j \to M$ and $\rho_j : M \to M_j$ for each $j \in I$ satisfying the following properties:

- $\rho_i \circ \mu^i = \mathrm{id}_{M_j}$ for each $I \in I$;
- $\rho_i \circ \mu^j = 0$ for each $i, j \in I, i \neq j$;
- $\rho_i(x) = 0$ for each $x \in M$, for almost all indices $i \in I$;[1]
- $\sum_{i\in I} \mu^i \circ \rho_i = \mathrm{id}_M$.

Exercise la.56$_\square$ An R-module M is called **free** if it is of the form $R^{(\Gamma)} = \sum_{\gamma\in\Gamma} R$, for a set Γ.

Find an R-module that is not free, when $R = \mathbf{Z}$; show that a $\mathbf{Z}$-module is free when 'it has a basis' in the sense of vector spaces (but be careful, it is possible to find rings where free modules do not have a well-defined dimension).

Exercise la.57$_\square$ Prove that if M is a free module over an infinite set of generators, then $M \cong M \oplus M$; deduce that there is an isomorphism between the abelian groups $\mathrm{End}(M)$ and $\mathrm{End}(M) \times \mathrm{End}(M)$. Is this isomorphism also a *ring* isomorphism?

Exercise la.58$_\square$ Let

$$\begin{tikzcd} & L \arrow[r, "u_1"] \arrow[d, "f"'] & M \arrow[r, "u_2"] \arrow[d, "g"'] & N \arrow[r, "u_3"] \arrow[d, "h"'] & 0 \\ 0 \arrow[r, "d_1"'] & L' \arrow[r, "d_2"'] & M' \arrow[r, "d_3"'] & N' & \end{tikzcd} \tag{4.34}$$

[1]A notation with which it's better to familiarise soon: *almost all* elements of a set means 'all, but possibly a finite number'.

be a diagram of F-linear maps between vector spaces, with the property that the rows are exact sequences (cf. Definition 4.4). Show that there exists an exact sequence

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \tag{4.35}$$

where $\operatorname{coker} f$ is the 'cokernel' of f, i.e. the quotient space $L'/\operatorname{im} f$, and similarly for $\operatorname{coker} g$ and $\operatorname{coker} h$.

Exercise LA.59$_\square$ The tensor product $V \otimes_s W$ of two super vector spaces $V = (V_0, V_1)$ and $W = (W_0, W_1)$ over the same field F is the usual vector space $V \otimes W$ equipped with the $\mathbf{Z}/2\mathbf{Z}$-graduation

$$(V \otimes W)_l = \sum_{i+j=l \mod 2} V_i \otimes W_j$$

. What is the universal property of this object?

Exercise LA.60$_\square$ Show that there exists a super vector space J acting as 'the square root of -1', in the sense that J is not the tensor unit and $J \otimes_s J \cong F$.

A **super algebra** over a field F is a super vector space V equipped with an F-bilinear multiplication operation: this means that there is an operation $V \times V \to V : (a, b) \mapsto a \cdot b$, such that $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b + c) = a \cdot b + a \cdot c$, and $(\alpha a) \cdot (\beta b) = \alpha\beta(a \cdot b)$ for each $a, b, c \in V$ and $\alpha \in F$).

Definition 4.5. Given a vector $v \in \mathbf{R}^2$ define a binary operation, the *Clifford norm*, as follows:

$$(a_1 e_1 + a_2 e_2) \bullet (a_1 e_1 + a_2 e_2) = a_1^2 e_1 \bullet e_1 + a_2^2 e_2 \bullet e_2 + a_1 a_2 (e_1 \bullet e_2 + e_2 \bullet e_1)$$

where (a_1, a_2) are the coordinates of v in the standard basis of $\mathbf{R}^2$. Now, if on this expression we impose the relation $v \bullet v = v \cdot v \in \mathbf{R}$ (where $v \cdot v$ is the scalar product of vectors), from the above equation we get that

$$e_i \bullet e_i = 1 \qquad\qquad e_1 \bullet e_2 = -e_2 \bullet e_1.$$

Define the **Clifford algebra** $Cl(2, \mathbf{R})$ as the set of elements of the form $a + b_1 e_1 + b_2 e_2 + \underline{c} e_{12}$, where $a \in \mathbf{R}$ is the *scalar part* of a Clifford vector x, $\vec{b} = (b_1, b_2) \in \mathbf{R}^2$ the *vector part* of x, and $\underline{c} \in \mathbf{R}$ its *bivector part* (where for the sake of brevity we write $e_{12} = e_1 \bullet e_2$); each of these three parts is a different *homogeneous component* of $x \in Cl(2, \mathbf{R})$.

Exercise LA.61$_\square$ The set $Cl(2, \mathbf{R})$ is a vector space, where the vector space operations are done componentwise (with the sum in $\mathbf{R}$ in the scalar and bivector part, and with the sum in $\mathbf{R}^2$ in the vector part). Prove that this is in fact a vector space.

Exercise LA.62$_\square$ Find an explicit formula for the Clifford product of two elements of $Cl(2, \mathbf{R})$:

$$(x + \vec{y} + \underline{z}) \bullet (x' + \vec{y}' + \underline{z}') = \dots \tag{4.36}$$

Exercise LA.63$_\square$ Prove that the Clifford product of two homogeneous elements of vector type, $v = (v_1, v_2), w = (w_1, w_2)$ sn't homogeneous any more, and in fact it decomposes as a nontrivial scalar part plus a nontrivial bivectorial part: the Clifford product of two vectors in $Cl(2,\mathbf{R})$ is

$$\vec{v} \bullet \vec{w} = \vec{v} \cdot \vec{w} + (\vec{v} \wedge \vec{w})e_{12}$$

where $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$ is the dot product of vectors and $\vec{v} \wedge \vec{w} = v_1 w_2 - v_2 w_1$ their cross product.

Exercise LA.64$_\square$ As a consequence of the previous exercise, prove the relations

- $\vec{v} \cdot \vec{w} = \frac{1}{2}(\vec{v} \bullet \vec{w} + \vec{w} \bullet \vec{v})$;
- $\vec{v} \wedge \vec{w} = \frac{1}{2}(\vec{v} \bullet \vec{w} - \vec{w} \bullet \vec{v})$;
- $\vec{a} \parallel \vec{b} \iff \vec{a} \wedge \vec{b} = 0 \iff \vec{a} \bullet \vec{b} = \vec{a} \cdot \vec{b}$;
- $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0 \iff \vec{a} \bullet \vec{b} = \vec{a} \wedge \vec{b}$.

Exercise LA.65$_\square$ Prove that the assignment sending $1 \mapsto \mathbf{I}_2$ (the 2×2 identity matrix), $e_1 \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $e_{12} \mapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ defines an algebra isomorphism θ between $Cl(2,\mathbf{R})$ and $M_2(\mathbf{R})$ (the algebra of 2×2 matrices with real coefficients); does this isomorphism depend on the choice we made for a basis of the two spaces?

Exercise LA.66$_\square$ Translate the matrix operations on $M_2(\mathbf{R})$ into operations in $Cl(2,\mathbf{R})$ along the isomorphism of LA.65:

- *transposition* of a matrix corresponds to changing the sign of the bivector part in the associated Clifford vector $\boldsymbol{u}$: if $\boldsymbol{u}$ and the matrix A correspond each other under θ, then A^t corresponds to the Clifford vector $\widetilde{\boldsymbol{u}} = u_0 + u_1 e_1 + u_2 e_2 - u_{12} e_{12}$;
- *inversion* of a matrix corresponds to taking the *Clifford conjugate* of $\boldsymbol{u}$: if $\boldsymbol{u}$ and the matrix A correspond each other under θ, and A is invertible, then A^{-1} corresponds to the Clifford vector $\overline{\boldsymbol{u}} = u_0 - u_1 e_1 - u_2 e_2 - u_{12} e_{12}$.

Exercise LA.67$_\square$ Prove that $\widetilde{(-)}$, $\overline{(-)}$ are involutive algebra antiautomorphisms: $\widetilde{\widetilde{\boldsymbol{u}}} = \boldsymbol{u} = \overline{\overline{\boldsymbol{u}}}$, $\widetilde{\boldsymbol{u}\boldsymbol{v}} = \widetilde{\boldsymbol{v}}\widetilde{\boldsymbol{u}}$, $\overline{\boldsymbol{u}\boldsymbol{v}} = \overline{\boldsymbol{v}}\,\overline{\boldsymbol{u}}$. How do $\widetilde{\overline{\boldsymbol{u}}}, \overline{\widetilde{\boldsymbol{u}}}$ relate to each other?

Exercise LA.68$_\square$ Show that $Cl(2,\mathbf{R})$ contains a subalgebra isomorphic to the field $\mathbf{C}$ of complex numbers, represented as the set of matrices

$$\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a,b \in \mathbf{R}\right\} \cong \{a + be_{12} \mid a,b \in \mathbf{R}\}$$

with real entries; in fact, e_{12} behaves like the imaginary unit, in that there exists a decomposition $Cl(2,\mathbf{R}) \cong \Big(\mathbf{R} \oplus e_{12}\mathbf{R}\Big) \oplus \Big(e_1\mathbf{R} \oplus e_2\mathbf{R}\Big)$ (prove it!).

Prove that this decomposition makes $Cl(2,\mathbf{R})$ a super algebra.

Part 2

Category theory

Short introduction. Category theory was born in 1945 with the work of Eilenberg and Mac Lane in algebraic topology, who wanted to axiomatise the situation when a family of functions

$$\alpha_X : A_X \to B_X$$

is constructed for each set X, and it varies accordingly to the presence of a function $f : X \to Y$ in the sense that the two functions α_X and α_Y are connected by a *path* like the following:

$$\begin{tikzcd} A_X \arrow[r] \arrow[d] & B_X \arrow[d] \\ A_Y \arrow[r] & B_Y \end{tikzcd}$$

Very rapidly, however, category theory invested the rest of Mathematics and contaminated abstract algebra, formal logic, differential and algebraic geometry, representation theory, touching to some deep extent even mathematical physics, probability theory, and combinatorics.

For a computer scientist, the most natural entry point to category theory is the following observation: in 1935, Gerhard Gentzen developed a profound approach to Hilbert's proof theory, in which formal laws for deriving logical entailments $A \vdash B$ (i.e., the premises in A lead to the conclusion in B) were carefully axiomatised. In 1940, Alonzo Church developed λ-calculus –and that move was considered very bold by many, as λ-calculus can be considered the first programming language ever invented. In 1962 William Howard noted, together with Haskell Curry, that these two things are essentially the same.

Category theory is a third thing, equal to both.

CHAPTER 5

Categories, functors, naturality

Exercise cf.1 $\square$ Verify that the following are examples of categories:

ce1) the empty category, having no objects and no morphisms; the category with a single object $\bullet$ and a single morphism $1_\bullet : \bullet \to \bullet$, playing the role of identity; the category with a single object $\bullet$, an identity morphism $1_\bullet$ and a non-identity morphism $e : \bullet \to \bullet$ subject to the relation $e \circ e = e$; the category with a single object $\bullet$, an identity morphism $1_\bullet$ and a non-identity morphism $e : \bullet \to \bullet$ subject to the relation $e \circ e = 1_\bullet$.

ce2) Set: objects are set, morphisms are functions between sets, composition is composition of functions, the identity of a given set A is the identity function $A \to A : a \mapsto a$.

ce3) $\mathsf{Alg}(T)$: A **signature** is a family $T = (n_i)_{i\in I}$ of natural numbers n_i, indexed by a set I. Let $T = (n_i)$ be a signature; define a category $\mathsf{Alg}(T)$ as follows: A T-algebra is a pair $(X, (t_i)_{i\in I})$ consisting of a set X and a family of functions $t_i : X^{n_i} \to X$, called n_i-ary operations on X. A T-homomorphism

$$f : \ (X, (t_i)_{i\in I}) \longrightarrow (Y, (s_i)_{i\in I}) \tag{5.1}$$

is a function $f : X \to Y$ for which the diagram

$$\begin{tikzcd} X^{n_i} \arrow[r, "f^{n_i}"] \arrow[d, "t_i"'] & Y^{n_i} \arrow[d, "s_i"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{5.2}$$

is commutative, meaning that for every operation $t_i : X^{n_i} \to X, s_i : Y^{n_i} \to Y$ and every n_i-tuple of elements $x_1, \dots x_{n_i}$ one has

$$f(t_i(x_1, \dots, x_{n_i})) = s_i(fx_1, \dots, fx_{n_i}) \tag{5.3}$$

Make precise the fact that as a corollary, all classes of algebraic structures form categories, when homomorphisms of structure are chosen as morphisms.

ce4) Rel: objects are sets; given sets A, B, the set of morphisms $A \to B$ is the powerset of $A \times B$, $\mathsf{Rel}(A, B) := 2^{A\times B}$. Given a relation $R \in 2^{A\times B}$ and a relation $S \in 2^{B\times C}$, define the composition $S \circ R \in 2^{A\times C}$ as the subset

$$(a, c) \in S \circ R \iff \exists b \in B,\ (a, b) \in R, (b, c) \in S. \tag{5.4}$$

Prove that this defines an associative composition operation.[1] Given this composition law, there is only one possible choice for the identity relation $I \subset A \times A$: find it. What relations, inside Rel, are *functional*? Argue that Rel contains the category Set of CE2.

CE5) Pred: the category of **predicates**, having objects the pairs (A, X) where A is a subset of X, and where morphisms $(A, X) \to (B, Y)$ are the functions $f : X \to Y$ such that $f(a) \in B$ for each $a \in A$;

CE6) Set_*: the category obtained from the previous one, restricting to the objects (A, X) for which A is a set with a single element. Make precise the statement that this is the category of 'pointed sets' and 'basepoint-preserving functions'.

CE7) (Various categories of spaces) There exist various categories having **topological spaces** as objects; a topological space is a set X equipped with a family of subsets $\tau \subseteq PX$ with the property that if $U, V \in \tau$ then $U \cap V \in \tau$ and if $U_* : I \to PX$ is any family $(U_i \mid i \in I)$ of subsets, each of which is an element of τ then $\bigcup U_i \in \tau$. Define a category Top of topological spaces with the following choice of morphisms:

> A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is a **homomorphism of topological spaces** (or a *continuous function* for short) if for every $V \in \tau_Y$ the set $f^{\leftarrow}V = \{x \in X \mid fx \in V\}$ belongs to τ_X.

Investigate whether the following alternative definition gives rise to a category, or exhibit the precise reason why it does not:

> A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is an **open map** if for every $U \in \tau_X$ the set $fU = \{fx \mid x \in U\}$ belongs to τ_Y.

CE8) Par: the category having objects sets, and morphisms $A \to B$ the functions $f : A \to B$, possibly defined only on a subset D of A (the 'domain').[2] How does this request affect the composition operation (the usual function composition) and the choice of identity morphisms? Make precise the statement that the categories Par and Set_* 'look very much alike'.

CE9) Σ-Seq: the category of (sequential) Σ-*acceptors*, where Σ is a finite set of input symbols, $\Sigma = \{\sigma_1, \dots, \sigma_n\}$. An object of Σ-Seq is a quadruple (Q, δ, q_0, F), where Q is a finite set of states, $\delta : \Sigma \times Q \to Q$ is a transition map, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states.

A morphism $f : (Q, \delta, q_0, F) \to (Q', \delta', q'_0, F')$ (called a simulation) between Σ-acceptors is a function $f : Q \to Q'$ that preserves

[1] If C is a class of sets, we say that a family of functions $\{f_{XYZ} : X \times Y \to Z\}$ indexed by the elements $X, Y, Z \in C$ is *associative* if

$$f_{WZU}(w, f_{XYZ}(x, y)) = f_{ZYU}(f_{WXZ}(w, x), y)$$

for every tuple X, Y, Z, U, W and elements for which this is meaningful. In the present case this evidently translates into the familiar associativity of composition $u \circ (v \circ w) = (u \circ v) \circ w$.

[2] For example, the function $f : \mathbf{R} \to \mathbf{R} : x \mapsto \frac{x+1}{x-1}$ is a morphism in Par, but not in Set.

- transitions, i.e., $\delta'(\sigma, f(q)) = f(\delta(\sigma, q))$,
- the initial state, i.e., $f(q_0) = q'_0$, and
- the final states, i.e., $f[F] \subseteq F'$.

CE10) Aut: the category of *automata*, with objects all (deterministic, sequential, Moore) automata. Objects are sextuples $(Q, \Sigma, Y, \delta, q_0, y)$, where Q is the set of states, Σ and Y are the sets of input symbols and output symbols, respectively, $\delta : \Sigma \times Q \to Q$ is the transition map, $q_0 \in Q$ is the initial state, and $y : Q \to Y$ is the output map. Morphisms from an automaton $(Q, \Sigma, Y, \delta, q_0, y)$ to an automaton $(Q', \Sigma', Y', \delta', q'_0, y')$ are triples (f_Q, f_Σ, f_Y) of functions $f_Q : Q \to Q'$, $f_\Sigma : \Sigma \to \Sigma'$, and $f_Y : Y \to Y'$ satisfying the following conditions:
- preservation of transition: $\delta'(f_\Sigma(\sigma), f_Q(q)) = f_Q(\delta(\sigma, q))$,
- preservation of outputs: $f_Y(y(q)) = y'(f_Q(q))$,
- preservation of initial state: $f_Q(q_0) = q'_0$.

CE11) turn these ideas into precise statements: (1) every poset $(P, \leq)$ gives rise to a category $c[P]$ with objects the elements of P, and where there is a unique morphism $x \to y$ if and only if $x \leq y$ in P; (2) every monoid M gives rise to a category $\mathbf{B}M$ having a single object $\bullet$, and where $\mathbf{B}M(\bullet, \bullet) = M$.

CE12) Define the category Vec_K where the set of objects is the set of natural numbers $\{0, 1, \dots, \}$ and the set of morphisms $n \to m$ is the set of $m \times n$ matrices with entries in the field K. In what sense this category 'does not lose information' contained in the (large) category of vector spaces?

CE13) Let Fld be the category of fields; starting from it we can define a category Vec (note the absence of subscript) containing *literally all* vector spaces in the following way: an object of Vec is a pair (K, V), where K is a field and V a vector space on K. A morphism $(K, V) \to (L, W)$ is a pair $u : K \to L$ and $f : V \to W$ such that u is a homomorphism of rings, and $f : V \to W$ a function such that $f(v + v') = fv + fv'$ and $f(av) = u(a)f(v)$ for every $a \in K, v \in V$. Define identities and compositions 'in the obvious way' and prove that the resulting structure is indeed a category.

CE14) Define a category C with a single object $\bullet$, and where the set of morphisms $\bullet \to \bullet$ is specified in BNF as

$$t ::= x_0 \mid c \mid f\,t \mid g\,t \tag{5.5}$$

where x_0 is one given variable, c is a constant and f, g are two *different* given function symbols. Composition is defined as substitution $t[t'/x_0]$ (where t' replaces x_0 in t is defined recursively).

If you know what to do with it, you are allowed to use a proof-assistant that checks the axioms of category.[3]

[3] If you are very brave: a monoid is exactly a category with a single object; this means that C above is isomorphic to a certain monoid M, whose elements are the terms $t = x_0 \mid c \mid f\,t \mid g\,t$ and whose monoid operation is defined by substitution. Describe the monoid M.

CE15) From the category of sets, remove all functions $A \to B$ when $A \neq B$ are different sets; is the result still a category C?

CE16) Recall that an **homotopy** between two continuous functions $f, g : X \rightrightarrows Y$ consists of a continuous function $H : X \times [0,1] \to Y$ with the property that for each $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$.
- Show that 'being homotopic' is an equivalence relation on the set of continuous functions $\mathsf{Top}(X,Y)$ (showing transitivity of the relation is, in particular, a delicate point); we denote the relation 'being homotopic' as $\simeq_{X,Y}$ or simply as $\simeq$.
- Given continuous maps $f, g \in \mathsf{Top}(X,Y)$ and $h \in \mathsf{Top}(A,X)$, show that if $f \simeq_{X,Y} g$ then $f \circ h \simeq_{A,Y} g \circ h$; similarly, if $k \in \mathsf{Top}(Y,B)$ and $f \simeq_{X,Y} g$, then $k \circ f \simeq_{X,B} k \circ g$.

Exercise CF.2□ Let C be a category. Show that 'being isomorphic' is an equivalence relation on the set of objects of a category.

Exercise CF.3□ Let C be a category. Show that 'there exists a morphism between' is an order relation on the set of objects of a category (usually, not antisymmetric).

Exercise CF.4□ Let C be a category. Study the properties of the relation 'being parallel' on the set of morphisms of C.

Exercise CF.5□ Deduce from the previous exercise that there is a poset C^p associated to every category C; the poset C^p is called the **posetal reflection** of C. Show that every functor $C \to P$, where $(P, \leq)$ is a poset, defines a unique monotone function $C^p \to P$.

Exercise CF.6□ Let A be a set; show that there exist
- the 'minimal' category A^δ on A, where the objects are the elements of A, and only identity morphisms exist;
- the 'maximal' category A^χ on A, where the objects are the elements of A, and there exists *exactly one* morphism between any two objects.

Show that any two objects of A^χ are isomorphic.[4]

Exercise CF.7□ Detail the construction that gives the 'minimal' and 'maximal' category on a set A (i.e., 'regard a set A as a discrete category', and 'regard a set A as a caegory with *exactly one* morphism between *any* two elements').

Exercise CF.8□ Among many others, a procedure to build a category is to 'force a bunch of monoids to live together': take a family of monoids M_i indexed by a set I, and define a category $\biguplus M_i$ having objects the elements $i \in I$ and morphisms specified by

$$\hom(i,j) = \begin{cases} M_i & i = j \\ \varnothing & i \neq j \end{cases} \tag{5.6}$$

Prove that this is indeed a category.

[4]The notation A^χ stands for the *chaotic* (Gr. $\chi\acute{\alpha}o\varsigma$, a primordial state of Being where there is no distinction between elements) category on A.

Exercise cf.9□ Recall that a **(directed) graph** $\mathcal{G}$ consists of a pair of sets G_0, G_1 equipped with functions

$$s, t : G_1 \to G_0 \tag{5.7}$$

sending each *edge* (element of G_1) to a pair of *vertices* (elements of G_0); a directed graph $\mathcal{G}$ gives rise to a **free category**, obtained as follows:

- the set of objects of $F\mathcal{G}$ is the set of vertices G_0 of $\mathcal{G}$;
- the set of morphisms $v \to w$ between two vertices $v, w \in G_0$ is the set of all tuples

  $$(v, \vec{x}, w) = v \to x_1 \to x_2 \to \cdots \to x_n \to w \tag{5.8}$$

  with the convention that if $v = w$ and $n = 0$ the tuple is empty and equal to an element $()_v \in F\mathcal{G}(v, v)$.

  The composition operation in $F\mathcal{G}$ is defined as

  $$(u, \vec{y}, w) \circ (v, \vec{x}, u) = (v, \vec{x} \uplus_u \vec{y}, w) \tag{5.9}$$

  where $\vec{x} \uplus_u \vec{y} = x_1 \to \cdots \to x_n \to u \to y_1 \to \cdots \to y_m$ if $\vec{x} = (x_1 \to \cdots \to x_n)$ and $\vec{y} = (y_1 \to \cdots \to y_m)$. Show that this is in fact a category (for each $v \in G_0$, the element $()_v$ is the identity arrow in $F\mathcal{G}$ for the composition defined above; composition is associative; etc).

Exercise cf.10□ Define a **verbose category** to be a tuple $\mathcal{A} = (\mathcal{O}, \mathcal{M}, \mathrm{d}, \mathrm{c}, \circ)$ consisting of

- a class $\mathcal{O}$, called the class of $\mathcal{A}$-objects,
- a class $\mathcal{M}$, called the class of $\mathcal{A}$-morphisms,
- functions $\mathrm{d} : \mathcal{M} \to \mathcal{O}$ and $\mathrm{c} : \mathcal{M} \to \mathcal{O}$, assigning to each morphism its domain and codomain, and
- a function $\circ$ from $D = \{(f, g) \mid f, g \in \mathcal{M} \text{ and } \mathrm{d}(f) = \mathrm{c}(g)\}$ to $\mathcal{M}$[with $\circ(f, g)$ written $f \circ g$],

subject to the following conditions:

- If $(f, g) \in D$, then $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ and $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$.
- If (f, g) and (h, f) belong to D, then $h \circ (f \circ g) = (h \circ f) \circ g$.
- For each $A \in \mathcal{O}$ there exists a morphism e such that $\mathrm{dom}(e) = A = \mathrm{cod}(e)$ and
  - $f \circ e = f$ whenever $(f, e) \in D$, and
  - $e \circ g = g$ whenever $(e, g) \in D$.
- For any $(A, B) \in \mathcal{O} \times \mathcal{O}$, the class $\{f \in \mathcal{M} \mid \mathrm{dom}(f) = A, \mathrm{cod}(f) = B\}$ is a set.

Compare the definition of verbose category with that of category and determine in which sense these definitions can be considered 'equivalent'.

Exercise cf.11$_\square$ Consider the category C/A of arrows with a common codomain A, and where morphisms are given by commutative triangles

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & A & \end{tikzcd} \tag{5.10}$$

(more formally, C/A has as objects all the arrows $h : X \to A$, and $C/A(h, k)$ for $h : X \to A$ and $k : Y \to A$ consists of the subset of $C(X, Y)$ made of those $f : X \to Y$ such that $kf = h$.)

Does C/A has an initial object? Does it have a terminal object? Try to write down the definition of product of two objects h, k in C/A; try to write down the definition of coproduct of two object h, k in C/A. A product of h, k in C/A is called the **pullback** of *fibered product* of h, k in C.

Exercise cf.12$_\square$ Describe as precisely as possible the category C/A for every example of category in cf.1.

Exercise cf.13$_\square$ Let C be a category; define the **arrow category** $C^{\to}$ having

- objects all the morphisms $u : X \to Y$ of C;
- morphisms $\left[\begin{smallmatrix} X \\ u\downarrow \\ Y \end{smallmatrix}\right] \to \left[\begin{smallmatrix} A \\ v\downarrow \\ B \end{smallmatrix}\right]$ the commutative squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & A \arrow[d, "v"] \\ Y \arrow[r, "g"'] & B \end{tikzcd} \tag{5.11}$$

Define identities and composition in the obvious way.

Does $C^{\to}$ have an initial object? Does it have a terminal object? Write down the definition of product and of coprodcut in $C^{\to}$.

In what sense, if any, $(C^{\mathrm{op}})^{\to}$ is equivalent to $(C^{\to})^{\mathrm{op}}$?

Exercise cf.14$_\square$ Verify whether the following are examples of functors: for those where only the correspondence on objects is given, try to define the one on morphisms or show that there is none making them a functor.

EF1) the empty functor $F_! : \varnothing \to C$, where $\varnothing$ is the empty category, and C is any other category; show that there is only one such functor.

EF2) the identity functor $1 : C \to C$ of a given category C, acting as the identity function both on objects and on morphisms.

EF3) the constant functor $c_X : \mathcal{A} \to C$, sending all objects $A \in \mathcal{A}_o$ to $X \in C_o$, and all morphisms $A \to A'$ to the identity morphism of X.

EF4) the functor $d_X : \bullet \to C$ from the singleton category, 'choosing the object X' and its identity. (Explain what this means formally.)

EF5) Given a category C and a morphism $f \in C(C_0, C_1)$, the functor $m_f : \{0 \to 1\} \to C$ from the category with two objects and a single nonidentity arrow, sending 0 to C_0, 1 to C_1, and $0 \to 1$ to $f : C_0 \to C_1$.

EF6) Given any category C and object $A \in C_o$, the functor $C(A,-) : C \to \mathsf{Set}$ sending $X \in C_o$ to the set of morphisms $u : A \to X$, and each morphism $g : X \to Y$ to the function $C(A,X) \to C(A,Y)$ sending $u : A \to X$ to $g \circ u : A \to Y$. The functor $C(A,-)$ is called *corepresentable*. The objec A is called the *representing object* or the *representative* of the functor.

EF7) Given any category C and object $A \in C_o$, the functor $C(-,A) : C^{\mathrm{op}} \to \mathsf{Set}$ sending $X \in C_o$ to the set of morphisms $u : X \to A$, and each morphism $g : X \to Y$ to the function $C(Y,A) \to C(X,A)$ sending $u : Y \to A$ to $u \circ g : X \to A$. The functor $C(-,A)$ is called *representable*. The objec A is called the *representing object* or the *representative* of the functor.[5]

EF8) Define a functor $\mathsf{Set} \to \mathsf{Mon}$ sending a set A to the set $A^\star$ of all finite lists with entries in A; how does a function $f : A \to B$ induces a monoid homomorphism $A^* \to B^*$?

EF9) Let G be a graph; send G to the set of its **connected components**, i.e. the set G_0 of its vertices modulo the equivalence relation generated by the source and target function $G_1 \to G_0 \times G_0$: $a, b \in G_0$ are equivalent if there is an edge $a \to b$ in G. Show that this is a functor $\mathsf{Graph} \to \mathsf{Set}$.

EF10) Try to define a functor $\mathsf{Grp} \to \mathsf{Ab}$ sending a group to its abelianization, i.e. the quotient of G by its commutator subgroup

$$[G,G] := \{aba^{-1}b^{-1} \mid a, b \in G\} \tag{5.12}$$

EF11) Try to define a functor $\mathsf{Grp} \to \mathsf{Ab}$ sending a group to its center.[6]

EF12) Try to define a functor sending a ring to its group of invertible elements.

EF13) There is a functor $\mathsf{Ring} \to \mathsf{Grp}$ sending a ring R to the group of $n \times n$ invertible matrices with entries in R. Define its correspondence on morphism and shot that it is a functor.

EF14) Send a ring R to the set of all its primes ideals. Is this a functor $\mathsf{Ring} \to \mathsf{Set}$? Covariant or contravariant?

EF15) Send a topological space to the set of its clopen (=both open and closed) subsets; is this a functor? Covariant or contravariant?

EF16) Send a group G to its **group algebra** $\mathbf{Z}[G]$, the set of formal sums of integers indexed by elements of G, $\sum_{g\in G} n_g$; define a ring operation on $\mathbf{Z}[G]$ and show that this is the object part of a functor $\mathsf{Grp} \to \mathsf{Ring}$. Recall that a few items ago you defined a functor sending a ring R to its group of invertible elements $R^\times$; construct a bijection between the set of group homomorphisms $G \to R^\times$ and the set of ring homomorphisms $\mathbf{Z}[G] \to R$.

EF17) Send a set X to the set of polynomials with integer coefficients $\mathbf{Z}[x \mid x \in X]$; does this define the object part of a functor $\mathsf{Set} \to \mathsf{Ring}$?

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[5]More generally, and in concordance with the principle of equivalence, a functor $F : C \to \mathsf{Set}$ is called corepresentable if there exists an object $A \in C$ and a natural isomorphism $C(A,-) \cong F$; similarly, $F : C^{\mathrm{op}}\mathsf{Set}$ is representable if there is a natural isomorphism $C(-,A) \cong F$ for some $A \in C$.

[6]The *center* of a group G is the set $\{g \in G \mid \forall x \in G,\ gx = xg\}$.

EF18) Send a ring R to the ring $R[t]$ of polynomials in a single indeterminate t; does this define the object part of a functor $\mathsf{Ring} \to \mathsf{Ring}$?

EF19) Send a group to the poset of all its subgroups; is this the object part of a functor $\mathsf{Grp} \to \mathsf{Pos}$?

EF20) Let M be a monoid regarded as a category with a single object. What is a functor $M \to \mathsf{Set}$? What is a functor $M^{\text{op}} \to \mathsf{Set}$?

EF21) Let $(P, \le)$ be a poset regarded as a category. What is a functor $(P, \le)^{\text{op}} \to \mathsf{Set}$?

EF22) In the notation above, let P be the poset of open subsets of the real line $\mathbf{R}$ (with respect to the usual 'Euclidean' topology). Consider the correspondence

$$U \longmapsto C^0(U) \tag{5.13}$$

sending an open subset $U \subseteq \mathbf{R}$ to the set of all continuous functions $f : U \to \mathbf{R}$. Show that this defines a functor $P^{\text{op}} \to \mathsf{Set}$, and in particular that every inclusion $U \subseteq V$ of open sets gives rise to a *restriction* operation $C^0V \to C^0U$ sending a function $f : V \to \mathbf{R}$ to its 'restriction' $f|_U : U \to \mathbf{R}$. Show that the following two properties are satisfied, given any $U \in P$ and any covering $\{V_i \mid i \in I\}$ of U:[7]

- if $f, g \in C^0(U)$ are such that $f|_i = g|_i$ for al $i \in I$, then $f = g$ ($f|_i = f|_{V_i}$ for short).
- if $f_i \in C^0(V_i)$ is a family of functions such that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$ for every $i, j \in I$, then there exists a function $f \in C^0(U)$ such that $f|_i = f_i$ for every $i \in I$.

EXERCISE CF.15□ Let C be a category. We say that C is **concrete** is there exists a faithful functor $U : C \to \mathsf{Set}$ from C to the category of sets and functions. Show that

- the categories of sets and functions, sets and relations, all categories of algebraic structures in the sense of CE3, the category of posets, the category of categories, the category of topological spaces and the categories of CE8, CE9, CE10, CE14 are all concrete;
- a category C is concrete if and only if its opposite category C^{op} is concrete;
- if a category C is concrete, all the slice categories C/X are concrete; is the converse true? (I.e.: if all slice categories are concrete, is C concrete?) More generally, if F, G are functors $\mathcal{A} \xrightarrow{F} C \xleftarrow{G} \mathcal{B}$, and $\mathcal{A}, \mathcal{B}$ are concrete, the comma category F/G is concrete;
- if C is concrete and $\mathcal{J}$ is a small category, the category of functors $F : \mathcal{J} \to C$ and natural transformations $\alpha : F \Rightarrow F'$ is concrete;

The variety of cases in which a category is concrete begs the question: is there a non-concrete category?

[7]A *covering* of $U \in P$ is a family $\{V_i \mid i \in I\}$ of elements of P such that $\bigcup V_i = U$.

Exercise cf.16□ Let $\mathcal{C}$ be a category, $X \in \mathcal{C}$ an object, and consider a morphism $f \in \mathcal{C}/X$; let $\mathcal{C}(f, B)$ be the set of morphisms $u, v : X \to B$ such that $u \circ f = v \circ f$, and define an equivalence relation on objects of $\mathcal{C}/X$ as follows:

$f \asymp g$ iff $\mathcal{C}(f, B) = \mathcal{C}(g, B)$ for every $B \in \mathcal{C}$,

The **Isbell criterion** characterizes concreteness in terms of the smallness of the quotient of $\mathcal{C}/X$ under the equivalence relation $\asymp$:

If $\mathcal{C}$ is concrete, then for every object $X \in \mathcal{C}$ the quotient of the class $(\mathcal{C}/X)_o$ under the equivalence relation $\asymp$ is a set.

Exercise cf.17□ Define a correspondence $G : \mathcal{C} \to \mathsf{Mon}$ (the category of monoids), sending a set A to the monoid $A^\star$ of finite lists of elements of A, i.e. to the set of all finite lists $(a_1, \dots, a_n)$ where $n \geq 0$ and $a_i \in A$ for each $i = 1, \dots, n$, and a function $f : A \to A$ to the function

$$A^\star \to A^\star : (a_1, \dots, a_n) \mapsto (f(fa_1), \dots, f(fa_n)) \tag{5.14}$$

Is G a functor $\mathcal{C} \to \mathsf{Mon}$?

Exercise cf.18□ Given two functors $F : \mathcal{C} \to \mathcal{X}$ and $G : \mathcal{D} \to \mathcal{X}$ define the **comma category** (F/G) having

- objects the triples (C, D, h) where $h : FC \to GD$ is a morphism in $\mathcal{X}$;
- morphisms $(C, D, h) \to (C', D', k)$ the pairs $u : C \to C'$, $k : D \to D'$ such that the square

$$\begin{tikzcd} FC \arrow[r, "Fu"] \arrow[d, "h"'] & FC' \arrow[d, "k"] \\ GD \arrow[r, "Gv"'] & GD' \end{tikzcd} \tag{5.15}$$

is commutative.

Verify that it is a category; does (F/G) have an initial object? Does it have a terminal object? Fix an object X of the codomain of a given functor $F : \mathcal{C} \to \mathcal{D}$; then, define the category (F/X) as the comma between F and the functor $d_X : \bullet \to \mathcal{D}$ 'selecting' X.

Exercise cf.19□ Describe as precisely as possible the comma category (F/X) for each functor in cf.14.

Exercise cf.20□ A **simple polynomial** is a functor $F : \mathsf{Set} \to \mathsf{Set}$ that is defined from the following inductive rules:

sp1) the identity functor $X \mapsto X$ is a simple polynomial;
sp2) every constant functor $X \mapsto A$ is a simple polynomial;
sp3) the product $F \times G : X \mapsto FX \times GX$ of two simple polynomials is simple;
sp4) the coproduct $\coprod_{i \in I} F_i : X \mapsto \coprod_{i \in I} F_i X$ of an arbitrary number of simple polynomials is simple.

An example of a polynomial functor is $X \mapsto A \times X^3 + B \times X^2 + X + 1$, where $\times$ denotes cartesian product, and $+$ denotes coproduct; another example is a 'formal series functor' $X \mapsto \coprod_{i\in I} A_i \times X^{n_i}$ where n_i are natural numbers and $(A_i \mid i \in I)$ is an arbitrary family of sets.

An *arity function* consists of a set I equipped with a function $a : I \to \mathbb{N}$; the inverse image $a^{-1}n$ is the set of elements in I having 'arity' n.[8] Every arity function $a : I \to \mathbb{N}$ defines an *arity functor* as

$$F_a : X \mapsto \coprod_{i\in I} X^{a(i)} = \{(i, \underline{x}) \mid i \in I,\ \underline{x} \in X^{a(i)}\} \tag{5.16}$$

Show that the class of simple polynomials coincides with the class of arity functors (F_a is 'clearly' a simple polynomial: how does one define an arity associated to a given simple polynomial?)

Exercise cf.21$_\square$ Let $\boldsymbol{P} : \mathsf{Set} \to \mathsf{Set}$ be the correspondence that sends a set A to the *power set* $\boldsymbol{P}A$ of A, the set of all subsets $U \subseteq A$, and a function $f : A \to B$ to the function $Pf : \boldsymbol{P}A \to \boldsymbol{P}B$, that sends a subset $U \subseteq A$ to the *image*

$$f_*U := \{fu \mid u \in U\} \tag{5.17}$$

Similarly, let $\boldsymbol{d} : \mathsf{Set} \to \mathsf{Set}$ be the correspondence that sends A to $\boldsymbol{P}A$, but a function $f : A \to B$ to the function $\boldsymbol{P}B \to \boldsymbol{P}A$, that sends a subset $V \subseteq B$ to the *inverse image*

$$f^*V := \{a \in A \mid fa \in V\} \tag{5.18}$$

- Show that both $\boldsymbol{P}, \boldsymbol{d}$ are functors ($\boldsymbol{d}$ is contravariant, i.e. $\boldsymbol{d}(f \circ g) = \boldsymbol{d}g \circ \boldsymbol{d}f$); show that given subsets $U \in \boldsymbol{P}A, V \in \boldsymbol{P}B$ one has
$$f_*U \subseteq V \iff U \subseteq f^*V. \tag{5.19}$$
- What is a natural transformation $f^*f_* \Rightarrow 1_{\boldsymbol{P}A}$, regarding $\boldsymbol{P}A$ as a category? Show that for each $U \in \boldsymbol{P}A$, one has $U \subseteq f^*f_*U$: is this a natural transformation $f^*f_* \Rightarrow 1_{\boldsymbol{P}A}$?

Exercise cf.22$_\square$ Verify that the following are examples of natural transformations:

- The family of set functions
$$\eta_A : A \longrightarrow A + E \tag{5.20}$$
where E is a fixed set, defined as the embedding of A as first summand in the disjoint union $A + E$ (if $A + E$ is built as it is customary as $(A \times \{0\}) \cup (E \times \{1\})$, then $\eta_A(a) = (a, 0)$; in more type-theoretic notation, η_A is just the map in_1).
- The family of set functions $\nabla : E + E \to E$ defined sending $(e, 0) \mapsto e$ and $(e, 1) \mapsto e$.
- The family of maps
$$\mu_A : (A + E) + E \longrightarrow A + E \tag{5.21}$$

[8]The word 'arity' is a back-formation from the Latin adjectival numeral suffix *-arius*, used to form adjectives from nouns or numerals.

defined as

$$\mathsf{unpack}\, x \,\mathsf{as}\, [x \,\mathsf{in}\, A + E, \mathrm{in}_2\, x \,\mathsf{in}\, E]$$

or, in less type-theoretic notation, as the composition

$$(A+E)+E \xrightarrow{\sim} A+(E+E) \xrightarrow{A+\nabla} A+E \tag{5.22}$$

- Given a morphism $f : X \to Y$ of a category C, show that there exists a natural transformation

$$C(Y,-) \longrightarrow C(X,-) \tag{5.23}$$

between representable functors (cf. CF.14.EF6); similarly, there exists a natural transformation

$$C(-,X) \longrightarrow C(-,Y) \tag{5.24}$$

- The family of functions $s_X : X \to \boldsymbol{P}X$ (where $\boldsymbol{P}X$ is the powerset of X, cf. CF.21) defined sending an element $x \in X$ to the singleton $\{x\}$, and the family of functions $\mu_X : \boldsymbol{PP}X \to \boldsymbol{P}X$ defined as follows: if a typical element of $\boldsymbol{PP}X$ is a family of subsets of X, then $\mu_X(\{A_i\}) = \bigcup_i A_i$.
- Let G be a group, and G^{ab} its abelianization G^{ab}, cf. CF.14.EF10. There is a natural projection map $G \to G^{\mathrm{ab}}$ sending an element of G into the coset $x[G,G]$.
- Let again $\boldsymbol{P}X$ denote the powerset of X and let 2^X be the set of functions $X \to \{0,1\}$. Since Equation 1.3 we know that there exists a bijection $\sigma : \boldsymbol{P}X \to 2^X$ sending each subset $U \subseteq X$ to its indicator function χ_U, with inverse the map sending $\gamma : X \to 2$ to the subset $U = \gamma^{\leftarrow}1$. Show that $\sigma_X : \boldsymbol{P}X \to 2^X$ is the component at X of a natural transformation, which is *componentwise invertible*; as a consequence, $X \mapsto \boldsymbol{P}X$ and $X \mapsto 2^X$ are isomorphic functors.
- Let K be a field, $n \geq 1$ a fixed natural number, and define a pair of functors GL_n and M_n as follows:
  - $\mathrm{GL}_n : \mathsf{CRing} \to \mathsf{Grp}$, sending a (commutative) ring R to the group of $n \times n$ invertible matrices with entries in R;
  - $M_n : \mathsf{CRing} \to \mathsf{Ring}$, sending a (commutative) ring R to the ring of $n \times n$ matrices with entries in R.

  Then, let $(-)^* : \mathsf{CRing} \to \mathsf{Grp}$ the functor sending a (commutative) ring R to the group R^* of its invertible elements.

  Is the determinant function $\det : M_n(R) \to R$ the component at R of a natural transformation $M_n \to J$ (where J is the tautological inclusion functor $\mathsf{CRing} \to \mathsf{Ring}$)? What if instead we take $\det : \mathrm{GL}_n(R) \to R^*$?

  Is there a natural transformation $\mathrm{GL}_n \to M_n$ whose components are the obvious inclusions of the invertible matrices in all matrices?

Show that the functors GL_n, M_n are *representable*, i.e. that there exist suitable rings M, G with the property that $\mathrm{GL}_n(R) \cong \mathsf{CRing}(G,R)$ and $M_n(R) \cong \mathsf{CRing}(M,R)$ (natural isomorphisms of functors).

Exercise cf.23$_\square$ Natural transformations are very common, but destroying the naturality of a transformation is very easy: find a natural transformation between two functors $F, G : C \to \mathcal{D}$, change the value of one of its components, and prove that the result is not natural any more.

Yoneda and representability: expand and complete

Exercise cf.24$_\square$ Prove that the following functors are examples of co/representable (cf. cf.14.ef6 and ef7)

- The identity functor $\mathsf{Set} \to \mathsf{Set}$ is represented by the singleton set $\{*\}$; if you're upset about Set not being a small category, prove the analogous result for the tautlogical embedding $\mathsf{Fin} \to \mathsf{Set}$ from the category of finite sets and functions to the category of all sets and functions. Do not forget to prove that the bijection $X \cong \mathsf{Set}(*, X)$ you find is natural in X!
- The functor $U : \mathsf{Cat} \to \mathsf{Set}$ that sends a category C to its set of objects is represented by the terminal category $\{*\}$; do not forget to prove that the bijection $UC \cong \mathsf{Cat}(*, C)$ you find is natural!
- The functor $V : \mathsf{Cat} \to \mathsf{Set}$ that sends a category C to its set of arrows is represented by the category $\{0 \to 1\}$ with two objects and a single non identity morphism between them; do not forget to prove that the bijection $VC \cong \mathsf{Cat}(\{0 \le 1\}, C)$ you find is natural!
- Slightly more generally, the functor $D_n : \mathsf{Cat} \to \mathsf{Set}$ sending C to the set
$$\{A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n\} \tag{5.25}$$
of all n-tuples of composable morphisms in C is representable: who is the representing object?
- Can you find two distinct natural transformations $\alpha_0, \alpha_1 : U \Rightarrow V$? Can you find two distinct natural transformations $\alpha_0, \alpha_1 : V \Rightarrow U$? Can you find two distinct natural transformations $V \Rightarrow V$? Can you find all natural transformations $D_n \Rightarrow V$? [Hint: how many functors are there $\{0 \le 1 \le 2\} \to \{0 \le 1\}$ if we consider the ordered sets as categories?]
- Consider the category Top of topological spaces (cf. **??**) and continuous functions and define a functor $O : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ sending a topological space (X, τ) to the set τ of its open subsets (regarded as a subset of $PX = 2^X$); show that O is representable by the space made from the set $S = \{0, 1\}$ equipped with the topology $\{\varnothing, \{1\}, S\}$; this topology is called the *Sierpiński* topology. [Hint: given a continuous map $f : (X, \tau) \to (S, \sigma)$, such f determines by an open subset $f^{\leftarrow}1$ and its complement $f^{\leftarrow}0$. Vice versa, for any open subset U of X the characteristic function $\chi_U : X \to S$ is continuous if the codomain S has the Sierpiński topology. Do not forget to prove that the bijection $OX \cong \mathsf{Top}(X, S)$ is *natural* in X!]
- Let again Top be the category of topological spaces; define a *disconnection* of a space X as a pair of open subsets $U, V \subseteq X$ such that $U \cup V = X$ and $U \cap V = \varnothing$. In simple terms, a disconnection of X is a way to prove that X is disconnected. Prove that sending X to the set of all its disconnections

is a functor $\mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$, represented by the space $D = \{0, 1\}$ equipped with the *discrete* topology (=all subsets are open). [Hint: adapt the previous line of reasoning!] Deduce the following disconnection criterion: a topological space (X, τ) is connected if and only if the only continuous functions $f : X \to D$ are the constants $fx = 0$ for all $x \in X$ and $fx = 1$ for all $x \in X$.

- Let C be the category of rings and homomorphisms of rings (sending 1 to 1); define a functor Z sending a ring R to its underlying set of elements; show that Z is representable, by the ring of polynomials $\mathbf{Z}[t]$ of polynomials in one indeterminate t. Don't forget to prove naturality!

Exercise cf.25$_\square$ Let's explore representability criteria.

- Prove that if $F : C \to \mathsf{Set}$ is corepresentable, then F commutes with all colimits that exist in C;
- Prove that $F : C \to \mathsf{Set}$ is corepresentable if and only if the category of elements of F (cf. **??**) has an initial object (dualize: $F : C^{\mathrm{op}} \to \mathsf{Set}$ is representable if and only if $\mathcal{E}(F)$ has a terminal object).

Exercise cf.26$_\square$ Define the *category of elements* of a functor $F : C \to \mathsf{Set}$ as follows:

- The objects of $\mathcal{E}(F)$ are the pairs (C, x) where C is an object of C and $x \in FC$ is an element;
- The morphisms $(C, x) \to (C', y)$ consist of the morphisms $f : C \to C'$ such that $Ff(x) = y$.

Prove that $\mathcal{E}(F)$ is indeed a category, if composition $(C, x) \xrightarrow{f} (C', y) \xrightarrow{g} (C'', z)$ is defined as $(C, x) \xrightarrow{gf} (C'', z)$.

Prove that the category of elements of F can be identified to the comma category of the pair of functors

$$\begin{tikzcd} & \bullet \arrow[d] \\ C^{\mathrm{op}} \arrow[r] & {[C, \mathsf{Set}]}. \end{tikzcd} \tag{5.26}$$

Prove that $\mathcal{E}(F)$ has an initial object if and only if F is representable, i.e. there exists an object $X \in C$ together with a natural isomorphism $F \cong C(X, \_)$.

CHAPTER 6

Co/limits

Exercise cl.1$_{\square}$ Let C be a category and $F : \mathcal{I} \to \mathcal{C}$ be a constant functor, say $FI = C$ for every $I \in \mathcal{I}$ and $Ff = 1_C$ for every $f : I \to I'$. Is it true that, when it exists, the limit of F is also C? If not, find a counterexample (easy) and a general formula to express $\lim F$ (harder). Dualise to the case of colimits.

Exercise cl.2$_{\square}$ Let the category Dyn be defined by having

- objects the triples (X, f, x_0) where X is a set, $f : X \to X$ an endofunction, and $x_0 \in X$ an element;
- a morphism $(X, f, x_0) \to (Y, g, y_0)$ is a function $u : X \to Y$ with the property that $u(x_0) = y_0$ and that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & X \arrow[d, "u"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd} \tag{6.1}$$

  is commutative.

Explain in what sense the initial object of this category is the set $\mathbf{N}$ of natural numbers. Prove that $\mathbf{N}$ is a monoid *using the universal property only*, i.e. defining by induction the operation $\_ + \_ : \mathbf{N} \to \mathbf{N} \to \mathbf{N}$.

Exercise cl.3$_{\square}$ Define the following categories, show the category axioms, and unwind the definition of what is a terminal object in each of them.

- Let S be a set and $\{A_s \mid s \in S\}$ a collection of sets indexed by S. Define the category $\Pi(A_s \mid s \in S)$ as follows: an object consists of a pair $(Z, \boldsymbol{f} = \{f_s \mid s \in S\})$ where Z is a set and $\boldsymbol{f} = \{f_s : Z \to A_s\}$ is a family of functions indexed by S; a morphism $(Z, \boldsymbol{f}) \to (W, \boldsymbol{g})$ consists of a function $u : Z \to W$ such that $g_s \circ u = f_s$ for every $s \in S$:

$$\begin{tikzcd} & A_s & \\ Z \arrow[ur, "f_s"] \arrow[rr, "u"'] & & W \arrow[ul, "g_s"'] \end{tikzcd} \tag{6.2}$$

  Investigate in particular the edge cases: what if $S = \varnothing$? What if S is a singleton? What if S has two elements $\{a, b\}$?
- Let S, X, Y be sets and $\{f_s : X \to Y\}$ a collection of functions with the same domain and codomain, indexed by S. Define the category $\Gamma(f_s \mid s \in S)$ as follows: an object consists of a pair $(Z, u : Z \to X)$

with the property that $f_s \circ u = f_t \circ u$ for every $s, t \in S$; a morphism $(Z, u) \to (W, v)$ consists of a function $h : Z \to W$ with the property that $v \circ h = u$:

$$\begin{tikzcd} Z \arrow[r, "u"] \arrow[dr, "h"'] & X \arrow[r, shift left=2, "f_s"] \arrow[r, shift left=1] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & Y \\ & W \arrow[u, "v"'] & \end{tikzcd} \tag{6.3}$$

Investigate in particular the edge cases: what if $S = \varnothing$? What if S is a singleton? What if S has two elements $\{a, b\}$?

- Let S be a set, and $\{f_s : X_s \to Y\}$ a family of functions with the same codomain Y, indexed by S. Define the category $\Lambda(f_s \mid s \in S)$ as follows: an object consists of a pair $(Z, \{u_s : Z \to X_s\})$ where Z is a set and $\boldsymbol{u} = \{u_s : Z \to X_s\}$ is a family of functions indexed by S, with the property that the composition $f_s \circ u_s : Z \to X_s \to Y$ is independent from the index $s \in S$; a morphism $(Z, \boldsymbol{u}) \to (W, \boldsymbol{v})$ consists of a function $t : Z \to W$ such that $v_s \circ t = u_s$:

$$\begin{tikzcd} W \arrow[r, "v_s"] \arrow[d, "v_{s'}"'] \arrow[dr] \arrow[dr, phantom, "\vdots"] & X_s \arrow[d, "f_s"] \\ X_{s'} \arrow[r, "f_{s'}"'] & Y \end{tikzcd} \tag{6.4}$$

Exercise cl.4$\square$ Fix a set A. Consider the functor S_A sending a set X to the set $1 + (A \times X)$, whose elements are of two kinds: either the single element $\perp \in 1$, or an element $(a, x) \in A \times X$.

An S_A-**algebra** consists of a pair (X, σ) where X is a set and $\sigma : S_A X \to X$ is a function. A morphism of S_A-algebras $(X, \sigma) \to (Y, \tau)$ consists of a function $u : X \to Y$ such that the square

$$\begin{tikzcd} 1 + (A \times X) \arrow[r, "\sigma"] \arrow[d, "1+A\times u"'] & X \arrow[d, "u"] \\ 1 + (A \times Y) \arrow[r, "\tau"'] & Y \end{tikzcd} \tag{6.5}$$

is commutative. Show that this defines a category $\mathsf{Alg}(S_A)$.

Describe the initial object A^* of $\mathsf{Alg}(S_A)$.

Exercise cl.5$\square$ Consider again the functor S_A defined above; an S_A-**coalgebra** consists of a pair (U, r) where U is a set and $r : U \to S_A U$ is a function. A morphism of S_A-coalgebras $(U, r) \to (V, t)$ consists of a function $h : U \to V$ such that the square

$$\begin{tikzcd} U \arrow[r, "r"] \arrow[d, "h"'] & S_A U \arrow[d, "S_A h"] \\ V \arrow[r, "t"'] & S_A V \end{tikzcd} \tag{6.6}$$

is commutative.

- Show that this defines a category $\mathsf{coAlg}(S_A)$;
- describe the terminal object $\hat{A}$ of $\mathsf{coAlg}(S_A)$;
- is there a relation between A^* and $\hat{A}$? (For example, can one be identified with a subset of the other?)

Exercise cl.6$_\square$ Let Ab be the category of abelian groups; find explicit descriptions for

- the product of two objects A, B; from this, derive an explicit description for $A_1 \times \cdots \times A_n$ for every $n \geq 2$;
- the coproduct of two objects A, B; from this, derive an explicit description for $A_1 + \cdots + A_n$ for every $n \geq 2$. In particular, prove that there is a natural isomorphism
$$A \times B \cong A + B \tag{6.7}$$
i.e. an isomorphism of functors $\_ \times \_ \cong \_ + \_$; the construction $A \times B \cong A + B$, in this context, is denoted $A \oplus B$ and called the **biproduct** of A, B;
- the equalizer $E(f, g)$ of a pair of homomorphisms $f, g : A \to B$; in particular, find an explicit description when $g = 0$ is the zero map; the equaliser of $(f, 0)$ is called the **kernel** of f;
- the pullback $A \times_C B$ of a pair of maps $A \xrightarrow{f} C \xleftarrow{g} B$; in particular, find an explicit description for the equalizer of the pair of maps
$$\mathbf{Z} \xrightarrow{\_\cdot m} \mathbf{Z} \xleftarrow{\_\cdot n} \mathbf{Z} \tag{6.8}$$

Exercise cl.7$_\square$ Prove that the pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is canonically isomorphic to the equaliser of the pair of maps

$$\begin{tikzcd} & C \arrow[d, "\Delta"] \\ A \oplus B \arrow[r, "f \oplus g"'] & C \oplus C \end{tikzcd} \tag{6.9}$$

where Δ is the diagonal map $x \mapsto (x, x)$.

Prove that the equaliser of $f, g : A \to B$ is canonically isomorphic to the kernel of the map $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] : A \oplus A \to B : a \mapsto f(a) - g(a)$.

Definition 6.1 (Semiadditive category). A category C admits a zero object if it has an initial object $\perp$, a terminal object $\top$, and the unique morphism $\perp \to \top$ is invertible.

A **semiadditive category** is a category C with a zero object, finite products and finite coproducts, such that the canonical map
$$A + B \longrightarrow A \times B \tag{6.10}$$
is the component at (A, B) of a natural isomorphism of functors $\_ \times \_ \cong \_ + \_$.

The category of abelian groups is semiadditive; more generally, the category of R-modules over a ring R is semiadditive (cf. la.51, la.52, la.53).

Exercise cl.8□ Show that in a semiadditive category $\mathcal{C}$, every set $\mathcal{C}(X, Y)$ becomes a commutative monoid under the operation

$$f + g : \; X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y. \tag{6.11}$$

Who is the identity element $0 : X \to Y$?

From here on, until cl.22 we will consider an explicit way to build pushouts in the category of sets. Let A, B be sets, and $f : A \to B$ be a function between them.

Exercise cl.9□ Show that f induces an equivalence relation $\approx_f$ on A as follows:

$$a \approx_f b \iff f(a) = f(b). \tag{6.12}$$

Exercise cl.10□ Let $f, g : A \to B$ be two functions. Show that f, g induce a relation on B as follows:

$$b \; R_{f,g} \; b' \iff \text{for some } a \in A,\; f(a) = b \text{ and } g(a) = b'. \tag{6.13}$$

Usually, $R_{f,g}$ is *not* an equivalence relation (why? Build a minimal counterexample); we denote $\approx_{f,g}$ the equivalence relation $\overline{R_{f,g}}$ generated by $R_{f,g}$.

Exercise cl.11□ Consider a function $f : A \to B$ and the equivalence relation $\approx_f$; let $Q = A/\approx_f$ be the quotient set, i.e. the set of equivalence classes $\{[a] \mid a \in A\}$, where $[a] := \{a' \in A \mid a \approx_f a'\}$. Let $\pi_f : A \to Q$ be the projection on the quotient set.

Prove that there exists a *unique* and *injective* function $\bar{f} : Q \to B$ such that the diagram

$$\begin{tikzcd} & A \arrow[dl, "\pi_f"'] \arrow[dr, "f"] & \\ Q \arrow[rr, dotted, "\bar{f}"'] & & B \end{tikzcd} \tag{6.14}$$

is commutative. [Hint: $\bar{f}$ must be defined sending $[a] \in Q$ to $f(a)$: prove that this is a well-defined function.]

More in general,

Exercise cl.12□ Let A be a set and R an equivalence relation on A; write $x \asymp y$ whenever $(x, y) \in R$.

A function $f : A \to X$ is said to be **compatible** with R (or *constant on the equivalence classes of* R) if

$$x \approx y \Rightarrow f(x) = f(y). \tag{6.15}$$

Prove the **first isomorphism theorem for sets**: given a set A with an equivalence relation R, whenever $f : A \to B$ is constant on the equivalence classes of R, there exists a unique $\bar{f} : A/R \to B$ such that

$$\begin{tikzcd} & A \arrow[dl, "\pi_f"'] \arrow[dr, "f"] & \\ A/R \arrow[rr, dotted, "\bar{f}"'] & & B \end{tikzcd} \tag{6.16}$$

is commutative.

Exercise cl.13$_\square$ (Density of R in $\overline{R}$.) Let R be a relation on a set A and $f : A \to B$ be a function. Similarly to cl.12, f is said to be *compatible with* R if for each $a, a' \in A$,

$$a \, R \, a' \quad \Rightarrow \quad f(a) = f(a'). \tag{6.17}$$

Prove (or disprove) that if f is compatible with R, then f is compatible with $\overline{R}$ (cf. 78).

The fundamental building block to understanding the construction of pushouts in Set is the equivalence relation generated by a pair of functions f, g.

Many problems in Mathematics involve identifying subspaces according to certain rules or performing quotients (think, for example, of how integers modulo n are usually defined).

One particular instance of such a situation is the following: we are given three sets (or spaces, or types...) A, B, C and functions f, g as follows,

$$B \xleftarrow{f} A \xrightarrow{g} C \tag{6.18}$$

and we are trying to build a third set/space P receiving maps from B, C and where each point in B of the form $f(a)$ gets identified with the point $g(a)$ in C. In other words, we want to construct a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & P \end{tikzcd} \tag{6.19}$$

subject to a certain universal property. We want the identification $fa \approx ga$ to happen "minimising" the quotient that we perform, or in other words, we want to identify $f(a)$ and $g(a)$ only and nothing else.

We want to perform this identification so that P possesses a universal property.

Definition 6.2. Let A, B, C be sets, and f, g be functions as follows:[1]

$$B \xleftarrow{f} A \xrightarrow{g} C \tag{6.20}$$

Define a category $\Pi(f, g)$ as follows:

- objects are all triples (X, p, q) where $p : C \to X$ and $q : B \to X$ are such that the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q"] \\ C \arrow[r, "p"'] & X \end{tikzcd} \tag{6.21}$$

  is commutative.

[1]A similar definition holds replacing sets A, B, C with the objects A, B, C of any category, and f, g with morphisms therein. We gladly leave this straightforward rephrasing to the reader.

- Morphisms $h : (X, p, q) \to (Y, p', q')$ are the functions $h : X \to Y$ with the property that $h \circ p = p'$ and $h \circ q = q'$, or in other words, $h : X \to Y$ fits in the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "q'"] & \\ C \arrow[r, "p"'] \arrow[drr, bend right, "p'"'] & X \arrow[dr, "h"] & \\ & & Y \end{tikzcd} \tag{6.22}$$

making both triangles commutative.

Exercise cl.14$_\square$ Define the identity morphisms $1_{(X,p,q)}$ in $\Pi(f, g)$, and composition using the category structure of Set. Prove that $\Pi(f, g)$ is a category.

Definition 6.3. The **pushout** of f, g (also called: the *cofibered sum* of B, C along A; the *amalgam* of B, C along f, g; the cocartesian product of f, g; etc.) is the initial object of $\Pi(f, g)$.

In many cases, the pushout of f, g is denoted as $B +_A C$.

Exercise cl.15$_\square$ Since $B +_A C$ is defined as the initial object of $\Pi(f, g)$, it must be unique up to a unique isomorphism; prove it.

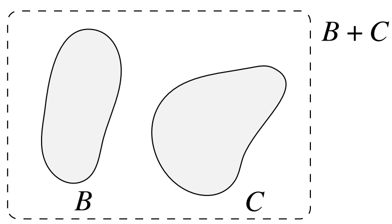
More precisely, prove the following statement:

> Define the category $\Pi(f, g)$ for any f, g as in (6.18); prove that if the pushouts $B +_A C$ exists, it is unique up to a unique isomorphism in $\Pi(f, g)$. (It's no more difficult to do this in Set than elsewhere!)

In the category of sets and functions, the pushout $B +_A C$ exists (we will build it explicitly in cl.18); in many other categories (for example, in categories of sets with algebraic structures: monoids, groups, abelian groups, vector spaces...) pushouts exist, but they are usually more difficult to describe explicitly than they are in Set.

Let us start to analyse pushouts in a situation where they reduce to an already known construction: when $A = \varnothing$, $f : \varnothing \to B$ and $g : \varnothing \to C$ consist of the unique functions from the empty set to B, C respectively (colloquially, the "initial maps" of B, C).

Recall the definition of the disjoint union or **coproduct** of B, C: it's the set $(B \times \{0\}) \cup (C \times \{1\})$, or more pictorially, the set obtained putting together B and C in a way that they have empty intersection:



(6.23)

Exercise CL.16$_\square$ Prove that the coproduct of B, C satisfies the following (two-fold) property: there exist functions $i_B : B \to B+C$ and $i_C : C \to B+C$ such that, for every other pair of functions

$$\begin{tikzcd} C \arrow[r, "p"'] & X & B \arrow[l, "q"] \end{tikzcd} \tag{6.24}$$

there exists a unique $u : B + C \to X$ making all parts of the following diagram commutative:

$$\begin{tikzcd} & B+C \arrow[d, "u"] & \\ C \arrow[ur, "i_C"] \arrow[r, "p"'] & X & B \arrow[ul, "i_B"'] \arrow[l, "q"] \end{tikzcd} \tag{6.25}$$

Exercise CL.17$_\square$ Prove that $B + C$ is the pushouts of f, g if $f : \varnothing \to B$ and $g : \varnothing \to C$ are the initial maps.

Unwinding the request, you have to prove that

- there exists a commutative diagram

$$\begin{tikzcd} \varnothing \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i_B"] \\ C \arrow[r, "i_C"'] & B+C \end{tikzcd} \tag{6.26}$$

- for every other commutative diagram like

$$\begin{tikzcd} \varnothing \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q"] \\ C \arrow[r, "p"'] & X \end{tikzcd} \tag{6.27}$$

  there exists a unique $u : B + C \to X$ such that $h \circ i_B = q$, $h \circ i_C = p$.

Notation**6.4**$_\square$. I order to stress the dependence of $u : B + C \to X$ above from p, q this function is usually denoted $\left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right]$ or, in a more type theore-y notation, as

$$\mathsf{unpack}\ u(z)\ \mathsf{as}\ [q(z)\ \mathsf{if}\ z \in B, p(z)\ \mathsf{if}\ z \in C]. \tag{6.28}$$

This roughly means the following: u acts on elements (is "computed") in different ways according to whether $z \in B + C$ belongs: if $z \in B$, compute $q(z)$, otherwise compute $p(z)$.

This little fact justifies the nomenclature: the pushout of f, g is an *amalgam* of B, C according to a rule specified by the structure of A and the definition of f, g; if A is empty, this rule is empty as well, and there is no identification to perform: no point of B is identified with any point of C, because the set of all elements of the form $f(a)$ is empty (as well as the set of all elements of the form $g(a)$).

Things get more interesting when A is allowed to be nonempty, and many known algebra and geometry constructions fall under this umbrella.

Let f, g be functions as in (6.18); consider the coproduct $P' = B + C$ and define a relation S on P' as follows: S contains all pairs of the form (fa, ga) for each $a \in A$.

Define $P = P'/\overline{S}$, the quotient of P' by the equivalence relation generated by S. As such, P has a canonical function from P', the projection to the quotient $\pi_S : P' \to P$.

Define the diagram

$$B \xrightarrow{i_B} P \xleftarrow{i_C} C \tag{6.29}$$

as follows: i_B is the composition $B \to B + C = P' \xrightarrow{\pi_S} P$; similarly, define $i_C : C \to P' \xrightarrow{\pi_S} P$.

Exercise cl.18$_\square$ (An explicit construction of pushouts in Set.) Show that (P, i_B, i_C) as defined above is the pushouts of f, g in the category of sets.

In order to prove the universal property, assume given an object (X, p, q) of $\Pi(f, g)$; unwinding the definition, this amounts to a commutative diagram as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q"] \\ C \arrow[r, "p"'] & X. \end{tikzcd} \tag{6.30}$$

This means that for each $a \in A$, $x = f(a)$ and $y = g(a)$ are such that $q(x) = p(y)$. Recall the definition of $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ above: the fact that if $x \approx_{f,g} y$ in $B + C$, then $qx = py$ can be rephrased as follows:

$$x \approx_{f,g} y \Rightarrow \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right](x) = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right](y) \tag{6.31}$$

so that by cl.11 and cl.12, $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ is constant on the equivalence classes of S and hence (cf. cl.13) on the equivalence classes of $\overline{S}$, and thus defines a unique function $v : P \to X$ sending $[z] \in P$ to the element computed as follows:

- choose a representative $z \in [z]$;
- if $z \in B \times \{0\} \subseteq B + C$, then $v([z]) = q(z)$; if $z \in C \times \{1\} \subseteq B + C$, then $v([z]) = p(z)$.

This is a well-defined function, thanks to the commutativity in (6.30).

Exercise cl.19$_\square$ Prove that, in the notation established so far, $v : P \to X$ is the *unique* function making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i_B"] \arrow[ddr, bend left, "q"] & \\ C \arrow[r, "i_C"'] \arrow[drr, bend right, "p"'] & P \arrow[dr, "v"] & \\ & & X \end{tikzcd} \tag{6.32}$$

This concludes the proof that P is the pushout of f, g, i.e. the initial object of the category $\Pi(f, g)$ defined in Definition 6.2.

Exercise cl.20$_\square$ (Laborious exercise.) Provide an explicit construction for pushouts in the category of abelian groups: given abelian groups A, B, C and group homomorphisms

$$B \xleftarrow{f} A \xrightarrow{g} C \tag{6.33}$$

define H as the subgroup of $B \oplus C$ generated by elements of the form $(fa, -ga)$ with $a \in A$.

Consider the quotient group $P = (B \oplus C)/H$; show that P is the pushout of f, g. [Hint: quotienting for the subgroup H so defined amounts to "killing" all elements of the form $fc - gc$ identifying them to zero; this in turn amounts (formally speaking) to identifying $fc \approx gc$.]

Exercise cl.21$_\square$ Use the result in the previous exercise to compute the pushouts that follow if C_n denotes the cyclic group of order n (or more concretely, the set of integers modulo n):

pab1) The pushout of

$$C_3 \xleftarrow{g} 0 \xrightarrow{f} C_3$$

if both f, g are the zero map $f(t) = 0, g(t) = 0$ for every t;

pab2) The pushout of

$$C_m \xleftarrow{\pi'} \mathbb{Z} \xrightarrow{\pi} C_n$$

if π, π' are the projection on the quotient $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$;

pab3) The pushout of

$$C_9 \xleftarrow{g} C_3 \xrightarrow{f} C_6$$

if $f(1) = 3, g(1) = 2$;

pab4) The pushout of

$$C_3 \longleftarrow C_2 \longrightarrow C_5$$

(how many homomorphisms $C_2 \to C_3$ and $C_2 \to C_5$ are there? –Hint: not many...; use item pab1.)

Exercise cl.22$_\square$ Prove that pushouts can be glued together: consider sets and functions in the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[r, "f'"] & B \\ D \end{tikzcd} \tag{6.34}$$

The pushout of $f' \circ f, g$ can be built as follows: first, build the pushout of f, g,

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[r, "f'"] \arrow[d, "q"] & B \\ D \arrow[r, "p"'] & P \end{tikzcd} \tag{6.35}$$

then build the pushout of f', q,

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[r, "f'"] \arrow[d, "q"] & B \arrow[d, "q'"] \\ D \arrow[r, "p"'] & P \arrow[r, "p'"'] & Q. \end{tikzcd} \tag{6.36}$$

Prove that $(Q, p' \circ p, q')$ is the pushout of $f' \circ f, g$.

In runic notation, the above result takes the mysterious form

$$Q = (A +_C D) +_A B \cong D +_C B. \tag{6.37}$$

EXERCISE CL.23$_\square$ Recall the definition of the category Dyn of (unpointed) **dynamical systems**:

- Objects are pairs (X, s) where $s : X \to X$ is a function on the set X;
- Morphisms $(X, s) \to (Y, t)$ are functions $f : X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "t"'] & Y \end{tikzcd} \tag{6.38}$$

  is commutative.

A dynamical system (X, s) is called **reversible** if $s : X \to X$ is an invertible function. A morphism between two reversible dynamical systems is just a morphism of dynamical systems.

Show that the inclusion functor $\mathsf{RevDyn} \hookrightarrow \mathsf{Dyn}$ admits a left adjoint, i.e. that for every morphism

$$f : (X, s) \longrightarrow (A, \sigma) \tag{6.39}$$

where (A, σ) is a reversible dynamical system, there exist

- A reversible dynamical system $(\bar{X}, \bar{s})$ with a map $u : (X, s) \to (\bar{X}, \bar{s})$ of dynamical systems;
- a *unique* $\bar{f} : \bar{X} \to A$ which is a morphism of dynamical systems, with the property that $\bar{f} \circ u = f$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & A \\ \bar{X} \arrow[ur, "\bar{f}"'] & \end{tikzcd} \tag{6.40}$$

  thus realising the isomorphism

$$\mathsf{Dyn}(X, A) \cong \mathsf{RevDyn}(\bar{X}, A). \tag{6.41}$$

Exercise cl.24$_\square$ Define the following category:

$$\begin{tikzcd} & 1 \arrow[d, shift left] \arrow[d, shift right] \\ 0 \arrow[r, shift left] \arrow[r, shift right] & 2 \end{tikzcd} \tag{6.42}$$

where the parallel arrows are called $\iota_0, \iota_1 : 1 \rightrightarrows 2$ and $\jmath_0, \jmath_1 : 0 \rightrightarrows 2$.

The **joint coequaliser** for two pairs of functions $i_0, i_1 : X_1 \rightrightarrows X_2$ and $j_0, j_1 : X_0 \rightrightarrows X_2$ consists of a colimit for a diagram $X : \mathcal{J} \to \mathsf{Set}$ of shape $\mathcal{J}$; this means that there is a diagram

$$\begin{tikzcd} & X_1 \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_0"] \\ X_0 \arrow[r, shift left, "j_0"] \arrow[r, shift right, "j_1"'] & X_2 \end{tikzcd} \tag{6.43}$$

where $X(\iota_0) = i_0, X(\iota_1) = i_1$, etc., and a morphism $t : X_2 \to C$ such that $ti_0 = ti_1$ and $tj_0 = tj_1$, and such that t is initial with respect to this property, i.e. for every other $x : X_2 \to Z$ such that $xi_0 = xi_1$ and $xj_0 = xj_1$ one has $x = \bar{x}t$ for a unique $\bar{x} : C \to X$.

Show that the joint coequaliser of $(i_0, i_1), (j_0, j_1)$ can be obtained as follows: start from the diagram (6.43) above, and consider the diagram

$$\begin{tikzcd} & X_1 \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_0"] & \\ X_0 \arrow[r, shift left, "j_0"] \arrow[r, shift right, "j_1"'] & X_2 \arrow[r] \arrow[d] & U \arrow[d] \\ & V \arrow[r] & P \end{tikzcd} \tag{6.44}$$

where $h : X_2 \to U$ is the coequaliser of (j_0, j_1), $k : X_2 \to V$ is the coequaliser of (i_0, i_1) and P is the pushout of (k, h).

Exercise cl.25$_\square$ An object C of a category $\mathcal{C}$ is called **tiny** if the functor $\mathcal{C}(C, \_)$ preserves all colimits.

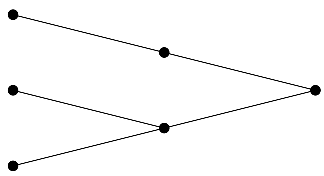
- Prove that the singleton $*$ of Set is a tiny object; is the coproduct of two tiny objects still tiny? Is the two-element set $* \coprod *$ tiny in Set? Prove that $\mathbf{Z}$ is a tiny object in the category of abelian groups. Is the group $\mathbf{Z}^n = \prod_{i=1}^n \mathbf{Z}$ still tiny?
- Prove that a functor $P \in [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ is a tiny object if and only if it is a retract of a representable functor.

Exercise cl.26$_\square$ (Transfinite constructions.) Consider the functor $F = 1 + \_ : \mathsf{Set} \to \mathsf{Set}$ sending a set A to the coproduct $1 + A$; consider the sequence

$$\varnothing \xrightarrow{u} F\varnothing \xrightarrow{Fu} FF\varnothing \xrightarrow{FFu} FFF\varnothing \tag{6.45}$$

or in other words the sequence $\varnothing \to \underline{1} \to \underline{2} \to \underline{3} \to \dots$, if $\underline{n}$ denotes the set $\{0, 1, \dots, n-1\}$. Show that the colimit of this sequence, regarded as a functor $(\omega, \leq) \to \mathsf{Set}$, is the set of natural numbers.

- What is the colimit of a similar sequence as in (6.45) (called the **initial chain** of F) when instead of F we consider the functor $F' : A \mapsto E + A$ where E is a fixed set of cardinality κ?
- What is instead the colimit of the initial chain for the functor $L : \mathsf{Set} \to \mathsf{Set} : A \mapsto 1 + A \times X$, where X is a fixed set? (Hint: for every $n \geq 1$, there is a simple expression for the nth iterate $L^n A = L \dots L\varnothing$.)
- Dualize the previous construction: compute the *limit* of the iterates of F, F' and L applied to the terminal morphisms $F1 \to 1$, $F'1 \to 1$ and $L1 \to 1$: the **terminal cochain** of F, for example, consists of the following diagram,



$$1 + 1 + 1 \xrightarrow{1+\nabla} 1 + 1 \xrightarrow{\quad\nabla\quad} 1$$

and the terminal cochain of L can be expressed as...

Exercise cl.27$_\square$ (The construction of coequalizers in Cat.) Consider two small categories $\mathcal{A}, \mathcal{B}$ and a pair of functors $F, G : \mathcal{A} \to \mathcal{B}$; suppose we have built the coequalizer C in

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \to C \tag{6.46}$$

and apply the functor $(-)_o : \mathsf{Cat} \to \mathsf{Set}$ sending a category to its set of objects: show that the diagram

$$\mathcal{A}_o \underset{G_o}{\overset{F_o}{\rightrightarrows}} \mathcal{B}_o \to C_o \tag{6.47}$$

remains a coequaliser in Set, and thus the object-class of C is precisely the quotient set of C_o modulo the minimal equivalence relation identifying the pairs of objets (Fa, Ga) (more precisely: the objects of C are the objects of $\mathcal{B}$ modulo the transitive closure of the relation $b \sim b' \iff \exists a : Fa = b, Ga = b'$). We denote equivalence classes $[b], [b']$ as Q, Q'.

The set $C([b], [b'])$ is now defined as follows: consider the set of finite words of contiguous morphisms in $\mathcal{B}$, i.e. all the finite sequences

$$b \overset{f_1}{\leftrightarrow} A_1 \overset{f_2}{\leftrightarrow} \cdots \overset{f_n}{\leftrightarrow} A_n \overset{f_{n+1}}{\leftrightarrow} b' \tag{6.48}$$

where an arrow can go in either direction, modded out with respect to the minimal equivalence relation $\asymp$ identifying the composition of adjacent morphisms, and the images of each $g \in \hom(\mathcal{A})$ via F, G: in other words, inductively define

$\asymp$ by posing $Fg \asymp Gg$ for all morphisms $g : b \to b'$ in $\mathcal{B}$ and declaring that $v \cdot F(g) \cdot w \asymp v' \cdot G(g) \cdot w'$ whenever $v \asymp v'$ and $w \asymp w'$.

Exercise cl.28$_\square$ Show that, defined in this way, C is a category: the identity morphism id_Q is $[\mathrm{id}_b]$ for any $b \in Q$, and this is well-defined because if $b \sim b'$ then $b = Fa, b' = Ga$ for some $a \in \mathcal{A}$, but then $F(\mathrm{id}_a) = \mathrm{id}_{Fa}$ and $G(\mathrm{id}_a) = \mathrm{id}_{Ga}$ get identified in the quotient. Composition of two paths is just concatenation of sequences, and it's easily seen to pass to the quotient in C (just reason by induction, first composing Fg, Fg', then Gg, Gg', then Fg, Gg; composing $v \cdot Fg \cdot w$ and $v' \cdot Gg \cdot w'$ uses the inductive hypotesis...)

Exercise cl.29$_\square$ Show that there is a functor $P \colon \mathcal{B} \to C$ given by "projection to the quotient" in the obvious way, which is such that every functor $H \colon \mathcal{B} \to C$ such that $HF = HG$ is constant on the generators of the equivalence relation $\asymp$. Thus, H uniquely factors through P, and the universal property is proved.

CHAPTER 7

Adjoints

Exercise AD.1 $\square$ (A very small example with many many adjoints **2**). Let $[2]$ be the category with two objects $0, 1$ and a single nonidentity arrow $0 \to 1$; more generally, let $[n]$ be the category

$$\{0 \to 1 \to \cdots \to n-1\} \tag{7.1}$$

(in this convention, $[1]$ is the terminal category, and $[0]$ is the empty category; such notation will soon come in handy; note that $[n]$ is the –unique up to monotone bijection– totally ordered set of cardinality n).

Show that there is a functor $e : [2] \to [1]$ that admits both a left and a right adjoint, $d_0 \dashv e \dashv d_1$. Do such adjoints admit adjoints in turn? More precisely, can you extend the string of adjoints $d_0 \dashv e \dashv d_1$ to a larger $l \dashv d_0 \dashv e \dashv d_1 \dashv r$?

Exercise AD.2 $\square$ Let $F \dashv G \dashv H$ be three functors with $G : \mathcal{A} \to \mathcal{B}$; prove that F is fully faithful if and only if H is fully faithful.

Exercise AD.3 $\square$ Define a category Set_i as follows:

- objects of Set_i are pairs $(A, e : A \to A)$ where A is a set and $e : A \to A$ is a function that is **idempotent**, i.e. $e \circ e = e$;
- morphisms $f : (A, e) \to (B, e')$ are functions $f : A \to B$ that 'commute with idempotents', i.e. such that for every $a \in A$, $e'(f(a)) = f(e(a))$.

Define an obvious 'forgetful' functor $U : \mathsf{Set}_i \to \mathsf{Set}$ sending each object (A, e) to A and $f(A, e) \to (B, e')$ to $f : A \to B$. Does U have a left adjoint, a right adjoint? Compute them explicitly, and address again the problem of extension above: if U has a left adjoint L, does L admit a left adjoint itself? If U has a right adjoint R, does R admit a right adjoint itself?

Exercise AD.4 $\square$ Let $\mathcal{C}$ be a category with products where every functor $A \times -$ has a right adjoint; prove that if $\mathcal{C}$ has a zero object (cf. Definition 6.1), then it must be the trivial category (with a single object and a single identity morphism).

Exercise AD.5 $\square$

- Disprove the false Cantor-Schröder-Bernstein theorem for categories: if there exists a faithful functor $F : \mathcal{C} \to \mathcal{D}$ and a faithful functor $G : \mathcal{D} \to \mathcal{C}$, there is an equivalence of categories $\mathcal{C} \cong \mathcal{D}$;
- prove the true Cantor-Schröder-Bernstein theorem for categories: if there exist a faithful functor $F : \mathcal{C} \to \mathcal{D}$ and a faithful functor $G : \mathcal{D} \to \mathcal{C}$, then $\mathcal{C}, \mathcal{D}$ must be equivalent.

Exercise ad.6□ Let $F \dashv G \dashv H$ be a triple of adjoints; show that $FG \dashv HG$ and $GF \dashv GH$. Let $H \dashv F \dashv G \dashv K$ be a quadruple of adjoints. Show that there are adjunctions $FH \dashv FG \dashv KG$ and $HF \dashv GF \dashv GK$; each of these generates, in turn, other 2 adjunctions $FHFG \dashv KGFG$ and $FGFH \dashv FGKG$, and... Define (inductively, if you want) the 'adjunction number' ℓ_n of a string $F_1 \dashv \cdots \dashv F_n$ to be the number of distinct adjunctions that the string generates. Find ℓ_{666}.

Exercise ad.7□ Let Δ be the category having objects the $[n] = \{0 \le 1 \le \cdots \le n-1\}$ defined above, and morphisms the monotone functions. Define maps

- $d^n_i : [n-1] \to [n]$ the only injective function missing the element $i \in [n]$ in its image;
- $s^n_j : [n+1] \to [n]$ the only surjective function assuming the value $j \in [n]$ twice.

Show that a morphism $\alpha : [n] \to [m]$ has a left adjoint, regarded as a functor, if and only if $\alpha(n) = m$; in such a case, $\alpha_L : i \mapsto \min\{j \in [n] \mid \alpha(j) \ge i\}$.

Deduce that for each $j \in [n]$ the function s^n_j has a left adjoint; dualise both statement: what is the condition on α such that it has a right adjoint? Does s^n_j has a right adjoint too?

Exercise ad.8□ Let P, Q be two partially ordered sets, and $f : P \leftrightarrows Q : g$ an adjunction (a **Galois connection**, cf. Definition 1.9). Prove that an adjunction

$$\begin{tikzcd}[P^{\mathrm{op}}, \mathsf{Set}] \arrow[r, shift left=4.5, "\mathrm{Lan}_f"] \arrow[r, shift right=1.5, "g^*"'] & {[Q^{\mathrm{op}}, \mathsf{Set}]} \arrow[l, shift right=1.5, "f^*"'] \arrow[l, shift left=4.5, "\mathrm{Ran}_g"]\end{tikzcd} \tag{7.2}$$

exists, and describe the way in which Σ_f and Π_g act on objects and morphisms.

Exercise ad.9□ Define a category $\mathsf{Adj}_\infty(C)$ whose objects are infinite strings of adjoints

$$\{F_\bullet\} : \cdots \dashv F_{-1} \dashv F_0 \dashv F_1 \dashv F_2 \dashv \cdots \tag{7.3}$$

between endofunctors of C, and whose morphisms are natural transformations $\eta : F_0 \to G_0$:

(1) prove that each η induces a family $\{\eta_k\}$ of natural transformations such that $\eta_{2n} : F_{2n} \to G_{2n}$, $\eta_{2n+1} : G_{2n+1} \to F_{2n+1}$;
(2) prove that if $C = \mathbf{Ab}$ is the category of abelian groups, $\mathsf{Adj}_\infty(C)$ has a zero object. Does it have finite products?

Exercise ad.10□ A **THC situation**[1] consists of a triple $\mathfrak{thc} = \{\otimes, \wedge, [\_, \_]\}$ of (bi)functors between three categories $\mathcal{S}, \mathcal{A}, \mathcal{B}$, defined by the adjunctions

$$\mathcal{B}(S \otimes A, B) \cong \mathcal{S}(S, [A, B]) \cong \mathcal{A}(A, S \wedge B). \tag{7.4}$$

[1]Although the chemical formula $C_{21}H_{30}O_2$ can describe multiple isomers, the term THC usually refers to the *Delta-9-THC* isomer with chemical name *(-)-trans-*Δ^9*-tetrahydrocannabinol*.

Show that the variances of these functors are uniquely determined by these three adjunctions: if $\otimes : \mathcal{S} \times \mathcal{A} \to \mathcal{B}$, then

$$\wedge : \mathcal{S}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{A} \qquad [\,\_\,,\,\_\,] : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{S}. \tag{7.5}$$

Show that starting from these functors $\mathfrak{thc} = \{\otimes, \wedge, [\,\_\,,\,\_\,]\}$ a new THC situation is defined as $\mathfrak{thc}_{I,J} = \{\boxtimes, \curlywedge, \langle\,\_\,,\,\_\,\rangle\}$, on the categories $\mathcal{S}^{I^{\mathrm{op}}\times J}, \mathcal{A}^I, \mathcal{B}^J$, for every pair of categories $I, J \in \mathsf{Cat}$; if we put $F \boxtimes G \in \mathcal{B}^J$, starting from $F \in \mathcal{S}^{I^{\mathrm{op}}\times J}, G \in \mathcal{A}^I$, then

$$\mathcal{B}^J(F \boxtimes G, H) \cong \mathcal{S}^{I^{\mathrm{op}}\times J}(F, \langle G, H\rangle) \cong \mathcal{A}^I(G, F \curlywedge H). \tag{7.6}$$

Exercise ad.11$_\square$ Show that adjunctions have a natural action on THC situations: from $\mathfrak{thc} = \{\otimes, \wedge, [\,\_\,,\,\_\,]\}$ between categories $\mathcal{S}, \mathcal{A}, \mathcal{B}$, and given a triple of adjoints

- $\hat{S} \dashv S \dashv \check{S}$ with $S : \mathcal{S} \to \mathcal{S}'$;
- $\hat{A} \dashv A \dashv \check{A}$ with $A : \mathcal{A} \to \mathcal{A}'$;
- $\hat{B} \dashv B \dashv \check{B}$ with $B : \mathcal{B} \to \mathcal{B}'$;

we have that $\{B \circ (\hat{S} \otimes \hat{A}), S \circ (\hat{A} \wedge \check{B}), A \circ [\hat{S}, \check{B}]\}$ is a new THC situation on $\mathcal{S}', \mathcal{A}', \mathcal{B}'$.

Exercise ad.12$_\square$ (Isbell duality). Let $\mathcal{A}$ be a small category. Show that there is a pair of adjoint functors

$$O : \ [\mathcal{A}^{\mathrm{op}}, \mathsf{Set}] \rightleftarrows [\mathcal{A}, \mathsf{Set}]^{\mathrm{op}} \ : S \tag{7.7}$$

where O sends a functor $P : \mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$ to the functor defined as

$$(A \in \mathcal{A}) \mapsto [\mathcal{A}^{\mathrm{op}}, \mathsf{Set}](P, \mathcal{A}(-, A)) \tag{7.8}$$

$A \in \mathcal{A}$ to $[\mathcal{A}^{\mathrm{op}}, \mathsf{Set}](P, \mathcal{A}(-, A))$, and S sends a functor $Q : \mathcal{A} \to \mathsf{Set}$ to the functor sending $A \in \mathcal{A}$ to $[\mathcal{A}, \mathsf{Set}](Q, \mathcal{A}(A, -))$.

Exercise ad.13$_\square$ A **directed graph** is a covariant functor on the category $\Gamma = \{E \overset{s}{\underset{t}{\rightrightarrows}} V\}$. Show that the forgetful functor $U : \mathsf{dGph} \to \mathsf{Set}$, i.e. the functor sending a directed graph to its set of vertices, has both a left and a right adjoint.

Show that the left adjoint is given by the functor that sends a set X to the directed graph having X as set of vertices, and no edges. How is the right adjoint defined?

Exercise ad.14$_\square$ (A category where adjunctions are objects.) Let $F \,\frac{\eta}{\epsilon}\!\!\mid\, G : \mathcal{A} \rightharpoonup \mathcal{X}$ and $F' \,\frac{\eta'}{\epsilon'}\!\!\mid\, G' : \mathcal{A}' \rightharpoonup \mathcal{X}'$ be adjunctions; a **map** from the first adjunction to the second consists of a pair of functors $H : \mathcal{A} \to \mathcal{A}', K : \mathcal{X} \to \mathcal{X}'$ and a natural transformation α filling the square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow_\alpha"] & \mathcal{A}' \arrow[d, "F'"] \\ \mathcal{X} \arrow[r, "K"'] & \mathcal{X}' \end{tikzcd} \tag{7.9}$$

- Show that a map between the two adjunctions could have been equivalently defined as the same pair of functors H, K, plus a natural transformation

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\beta\Nearrow"] & \mathcal{A}' \arrow[d, "F'"] \\
\mathcal{X} \arrow[r, "K"'] & \mathcal{X}'
\end{tikzcd} \tag{7.10}$$

(Hint: define a correspondence from one kind of square to the other, 'pasting unit and counit':

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\Swarrow\alpha"] & \mathcal{A}' \arrow[d, "F'"] \\
\mathcal{X} \arrow[r, "K"'] & \mathcal{X}'
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
\mathcal{X} \arrow[r, "G"] \arrow[dr, equal, bend right, "\Swarrow\epsilon" description] & \mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\Swarrow\alpha"] & \mathcal{A}' \arrow[d, "F'"] \arrow[dr, equal, bend left, "\Swarrow\eta'" description] & \\
 & \mathcal{X} \arrow[r, "K"'] & \mathcal{X}' \arrow[r, "G'"'] & \mathcal{A}'
\end{tikzcd} \tag{7.11}$$

and show that it is bijective using the zig-zag identities.)
- When $\mathcal{A} = \mathcal{A}', \mathcal{X} = \mathcal{X}'$ and H, K are the respective identities, we call the natural transformations α, β corresponding to each other under the equivalence above a **conjugate pair**. Show that when α, β are conjugate in this sense, the diagrams

$$\begin{tikzcd}
FG' \arrow[r, "F\beta"] \arrow[d, "\alpha G'"'] & FG \arrow[d, "\epsilon"] \\
F'G' \arrow[r, "\epsilon'"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[d, "\eta'"'] & GF \arrow[d, "G\alpha"] \\
G'F' \arrow[r, "\beta F'"'] & GF'
\end{tikzcd} \tag{7.12}$$

commute.
- In fact, one can show a sharper result, that the following conditions are equivalent: i) α, β form a conjugate pair; ii) the left square above commutes; iii) the right square above commutes.
- Define functors

$$\mathsf{Cat}(\mathcal{A}, \mathcal{X}) \xleftarrow{\lambda} \mathsf{Adj}(\mathcal{A}, \mathcal{X}) \xrightarrow{\rho} \mathsf{Cat}(\mathcal{X}, \mathcal{A}) \tag{7.13}$$

as $(F \dashv G)^\lambda = F$ and $(F \dashv G)^\rho = G$. Study the properties of $(-)^\lambda$ and $(-)^\rho$: are they full, faithful, conservative?

Exercise AD.15$_\square$ (A category where adjunctions are morphisms.) Let $\mathcal{X}, \mathcal{Y}$ be categories. An **adjoint morphism** from $\mathcal{X}$ to $\mathcal{Y}$ consists of an adjunction $F \dashv G$ denoted

$$\begin{tikzcd}
\mathcal{X} \arrow[r, shift left, "F"] & \mathcal{Y} \arrow[l, shift left, "G"]
\end{tikzcd} \tag{7.14}$$

and we now define a *composition* operation between adjoint morphisms: given a diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, shift left, "F"] & \mathcal{Y} \arrow[l, shift left, "G"] \arrow[r, shift left, "F'"] & \mathcal{Z} \arrow[l, shift left, "G'"]
\end{tikzcd} \tag{7.15}$$

define a candidate adjunction $F'F \dashv GG'$ having unit

$$\eta \star \eta' := 1 \longrightarrow GF \xrightarrow{G\eta' F} GG'F'F \tag{7.16}$$

and counit

$$\epsilon' \star \epsilon := F'FGG' \xrightarrow{F'\epsilon G} F'G' \longrightarrow 1 \tag{7.17}$$

Let $\mathsf{Adj}(\mathcal{X}, \mathcal{Y})$ be the collection of adjoint morphisms from $\mathcal{X}$ to $\mathcal{Y}$; show that this defines an associative and unital composition operation

$$\begin{aligned} \mathsf{Adj}(\mathcal{Y}, \mathcal{Z}) \times \mathsf{Adj}(\mathcal{X}, \mathcal{Y}) &\longrightarrow \mathsf{Adj}(\mathcal{X}, \mathcal{Z}) \\ (F' \tfrac{\eta'}{\epsilon'}\big| G'), (F \tfrac{\eta}{\epsilon}\big| G) &\longmapsto (F'F \tfrac{\eta\star\eta'}{\epsilon'\star\epsilon}\big| GG',) \end{aligned} \tag{7.18}$$

Exercise ad.16$_\square$ Show that if

Adj is the prototype of a **2-category**, i.e. a structure having

- *objects* or 0-cells denoted by $X, Y, Z, \dots$;
- *morphisms* or 1-cells denoted by $f : X \to Y$;
- *transformations* or 2-cells denoted by $\alpha : f \Rightarrow g$

all together subject to the requirement that 1-cells $f : X \to Y$ form the objects of a category $\mathcal{K}(X, Y)$ whose morphisms are precisely the 2-cells α, and such that the composition operation

$$\_\#_0\_ : \mathcal{K}(X, B) \times \mathcal{K}(A, X) \longrightarrow \mathcal{K}(A, B) \tag{7.19}$$

is a bifunctor.

Exercise ad.17$_\square$ Show that the bifunctoriality of composition amounts to the request that

$$(\gamma\#_1\delta)\#_0(\alpha\#_1\beta) = (\gamma\#_0\alpha)\#_1(\delta\#_0\beta). \tag{7.20}$$

This is called the **interchange law**.

Exercise ad.18$_\square$ Prove that two functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ are adjoints, with F left adjoint to G, if and only if the two comma categories $(F/1_{\mathcal{D}})$ and $(1_{\mathcal{C}}/G)$ are 'equivalent over $\mathcal{C} \times \mathcal{D}$', namely there is an equivalence of categories $U : (F/1_{\mathcal{D}}) \cong (1_{\mathcal{C}}/G) : V$ with the property that the diagram

$$\begin{tikzcd} (F/1) \arrow[rr, dash] \arrow[dr, "X"'] & & (1/G) \arrow[dl, "Y"] \\ & \mathcal{C} \times \mathcal{D} & \end{tikzcd} \tag{7.21}$$

is commutative (choosing either U or its inverse V as horizontal arrow).

Here, $X : (F/1) \to \mathcal{C} \times \mathcal{D}$ is the functor that sends an object $(C, D, FC \to D)$ to the pair (C, D), and similarly Y sends an object $(C, D, C \to GD)$ to the pair (C, D).

Exercise ad.19$_\square$ Let $(\mathbf{Z}, \leq)$ be the totally ordered set of integers, regarded as a category, and $f : \mathbf{Z} \to \mathbf{Z}$ a monotone function, regarded as an endofunctor. Show that the following conditions are equivalent:

c1) f has a left adjoint f_L;

C2) f has a right adjoint f_R;
C3) the image $f(\mathbf{Z})$ of f is unbounded from below and from above.

(hint: show that f has a right adjoint if and only if the following condition holds:

D1) each set $S_m = \{n \mid fn \leq m\}$ is nonempty and bounded from above; thus $f_R(m) := \max S_m$.

Show that this, in turn, is equivalent to the third condition above. Dualise for left adjoints.)

Let $f : \mathbf{Z} \to \mathbf{Z}$ be the map sending an integer k to $2k$, so that the image of f is $2\mathbf{Z}$; what are the left (and the right) adjoints f_L, f_R of f?

Describe the monads obtained from the adjunction $f_L \dashv f$ and from the adjunction $f \dashv f_R$.

Exercise AD.20$_\square$ (Induced and coinduced G-representations). Let $i : H \leq G$ be the inclusion of a subgroup regarded as a functor between one-object categories; if k is a field, the category of k-linear representations of G can be identified with the functor category $[G, \mathsf{Vect}_k]$, and similarly for H. (Make this statement precise.)

The scope of this exercise is to show that the functor

$$[H, \mathsf{Vect}] \to [G, \mathsf{Vect}] \tag{7.22}$$

induced by precomposing with i (=restricting the action of G to the subgroup H) has both a left and a right adjoint, i.e. that every k-linear representation of H can be extended in a maximal and minimal way to a representation of the whole G.

- Show that there is an equaliser diagram

$$\hom_{k[H]}(k[G], V) \longrightarrow \hom_k(k[G], V) \rightrightarrows \textstyle\prod_{h \in H} \hom(k[G], V) \tag{7.23}$$

  where $k[G]$ is the group k-algebra of EF16, and the two parallel maps are obtained as follows: $l_h : \hom(k[G], V) \to \hom(k[G], V)$ is obtained sending $f : k[G] \to V$ to $f(h.\_)$, and $r_h : \hom(k[G], V) \to \hom(k[G], V)$ sending f to $h.f(\_)$.
- Show that there is a coequaliser diagram

$$\textstyle\coprod_{h \in H} K[G] \otimes_k V \rightrightarrows k[G] \otimes_k V \longrightarrow k[G] \otimes_{k[H]} V \tag{7.24}$$

  obtained quotienting by the relation prescribing $(h.\alpha) \otimes v - \alpha \otimes (h.v)$.

Exercise AD.21$_\square$ Let R, S be unital rings, and $f : R \to S$ an homomorphism. Define a functor

$$f^* : \mathsf{Mod}_S \longrightarrow \mathsf{Mod}_R \tag{7.25}$$

so that each S-module N is equipped with an action of R as follows: $r.n := f(r).n$; show that

- f^* has a left adjoint $f_!$ and describe its action on objects and morphisms;
- f^* has a right adjoint f_* and describe its action on objects and morphisms.

(Hint: let M be an R-module; show that S has a natural structure of R-module induced via f and that $f_!(M) \cong f^*S \otimes M$; dualize for f_*)

Exercise AD.22$_\square$ A notable result in the theory of adjoint functors is the **adjoint functor theorem**, establishing ufficient condition for the existence of a left adjoint to a functor $F : C \to \mathcal{D}$ that preserves all limits: a functor F that preserves all limits has a left adjoint if and only if it satisfies a certain condition known as the *solution set condition*; a key result towards the proof of this equivalence is the following **initial object lemma**:

> Let C be a category admitting all small limits; then, C has an initial object if and only if it has a **weakly initial family**, i.e. a set of objects $\{W_i \mid i \in I\}$ with the property that for every $X \in C$ there exists at least (but possibly many) arrow $W_{i(X)} \to X$.

Prove the initial object lemma, following this guide:

- If C has an initial object, it obviously has a weakly initial family;
- Conversely, build the product $W = \prod_{i\in I} W_i$ of all the elements of a weakly initial family.
- Consider the joint equaliser

$$K \xrightarrow{k} W \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; W \tag{7.26}$$

  of all endomorphisms of W (this means that k has the property that $ku = kv$ for every pair $u, v : W \to W$, and it is terminal with this property);
- K is a weakly initial object: why? Show that K is an initial object: assume $f, g : K \to X$ are parallel arrows out of K; show that $f = g$ (hint: $f = g$ if and only if their equaliser is isomorphic to K).

Exercise AD.23$_\square$ Let P, R be two posets; show that there is a partial order structure on the set R^P of all monotone functions $P \to R$ such that

$$\mathsf{Pos}(Q, R^P) \cong \mathsf{Pos}(P \times Q, R) \tag{7.27}$$

is an isomorphism natural in all its components P, Q, R.

Let $C, \mathcal{D}$ be categories. We define a **triple of adjoints** to be a triple of functors L, F, R such that

- $F : C \to \mathcal{D}$ and $L, R : \mathcal{D} \to C$;
- L is a left adjoint for F, and R is a right adjoint for F.

We denote a triple of adjoints with the following stenography:

$$L \dashv F \dashv R : C \xrightarrow{F} \mathcal{D}. \tag{7.28}$$

Exercise AD.24$_\square$

- Let τ, τ' be topologies on the same set X and suppose $\tau \subseteq \tau'$ (so, every open subset in τ is also open for τ'); show that there is a triple of adjoints

$$j_! \dashv j^* \dashv j_* : [\tau', \mathsf{Set}] \xrightarrow{j^*} [\tau, \mathsf{Set}] \tag{7.29}$$

  induced between categories of functors into Set.

- (compare with AD.20) Let G, H be two groups and $f : G \to H$ a homomorphism between them; show that there is a triple of adjoints

$$f_! \dashv f^* \dashv f_* : [H, \mathsf{Set}] \xrightarrow{j^*} [G, \mathsf{Set}] \tag{7.30}$$

induced between categories of functors into Set.

Exercise AD.25$_\square$ Let S be a set; define a category $B[S] = \{0 \xrightarrow{\{s\}} 1\}$ having two objects $0, 1$ and only non-identity morphisms in the set $S = B[S](0, 1)$; consider the diagram

$$\{\bullet\} \overset{d_0}{\underset{d_1}{\rightrightarrows}} B[S] \tag{7.31}$$

where $d_i : \bullet \mapsto i$. Describe explicitly the coequalizer (in the category Cat, cf. CL.27–CL.29) of (7.31) and the universal map $q : B[S] \to C$.

Exercise AD.26$_\square$ Show that there is an equaliser diagram

$$[C, \mathsf{Set}] \xrightarrow{q^*} [0 \rightrightarrows 1, \mathsf{Set}] \overset{d_0^*}{\underset{d_1^*}{\rightrightarrows}} \mathsf{Set} \tag{7.32}$$

where

- $q^* = [q, \mathsf{Set}]$ is the 'precomposition with q' functor $P \mapsto P \circ q$;
- we identify the category Gph of digraphs with the functor category $[0 \rightrightarrows 1, \mathsf{Set}]$;
- the category $[C, \mathsf{Set}]$ is the category of sets equipped with an action of the free monoids $M = \mathbb{N}\langle s, t\rangle$.

Definition 7.1. A *2-generated* monoid consists of a monoid M appearing as a quotient of $\mathbf{N}\langle s, t\rangle$, i.e. as a coequaliser diagram

$$N \rightrightarrows \mathbf{N}\langle s, t\rangle \longrightarrow M \tag{7.33}$$

Exercise AD.27$_\square$ Show that q^* is injective on objects and faithful, but not full. What are the morphisms in the image of $q^*_{XY} : [C, \mathsf{Set}] \to \mathsf{Gph}$?

Show that for every 2-generated monoid we get a triple of adjoints in a similar fashion as before.

Let $\mathsf{Gph} = [0 \rightrightarrows 1, \mathsf{Set}]$ be the category of digraphs and $\mathsf{aNet} = [\mathbf{N}\langle s, t\rangle, \mathsf{Set}]$ the category appearing in the coequaliser (7.32).

Exercise AD.28$_\square$ Prove that there exists a functor $D : \mathsf{Gph} \to \mathsf{aNet}$ defined as follows:

- on objects sends a graph $\boldsymbol{K} = (K_0, K_1, s, t)$ to the set $K_0 + K_1$ equipped with the action

$$K_0 + K_1 \overset{\sigma}{\underset{\tau}{\rightrightarrows}} K_0 + K_1 \tag{7.34}$$

where $\sigma = [i_1, i_1 \circ s]$ and $\tau = [i_1, i_1 \circ t]$;
- on morphisms of graphs $f : \boldsymbol{K} \to \boldsymbol{H}$, it acts as the bifunctor $\_ + \_$ sending (f_0, f_1) to $[f_0, f_1] : K_0 + K_1 \to H_0 + H_1$.

Exercise AD.29$_\square$ Prove that the functor D defined in AD.28 has a right adjoint.

Dualise the previous couple of results:

Exercise AD.30$_\square$ Define a functor $W : \mathsf{Gph} \to \mathsf{aNet}$; instead of taking a graph $\boldsymbol{K}$ to the coproduct of edges and vertices, let's define the carrier of $W\boldsymbol{K}$ to be $K_0 \times K_1$, and two functions $\sigma, \tau : W\boldsymbol{K} \to W\boldsymbol{K}$ as follows:

$$K_0 \times K_1 \overset{\sigma}{\underset{\tau}{\rightrightarrows}} K_0 \times K_1 \tag{7.35}$$

where $\sigma = \langle \pi_1, s \circ \pi_1 \rangle$ and $\tau = \langle \pi_1, t \circ \pi_1 \rangle$ if $s, t : K_1 \to K_0$ are the source and target functions of the graph.

Fill in the details of this construction; prove that W has a left adjoint, dualizing AD.29.

Exercise AD.31$_\square$ Show that there exists a quadruple of adjoint functors

$$\pi_0 \dashv d \dashv (-)_o \dashv c \tag{7.36}$$

where $\pi_0 : \mathsf{Cat} \to \mathsf{Set}$ is the functor sending a category C to its set of *connected components*, i.e. to the coequaliser

$$\hom(C) \rightrightarrows C_o \longrightarrow \pi_0(C). \tag{7.37}$$

CHAPTER 8

Monads