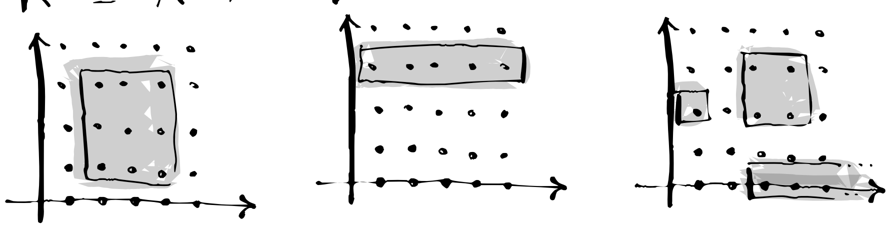
Ordered structures

we have introduced the notion of ordered set in order to axiomatise the idea of "precedence" between elements in a set A

<u>Definition</u> A **RELATION** on a set A consists of the choice of some pairs of elements of A that are called **COMPARABLE**

More formally, a relation (on a set A) is a subset $R \subseteq A \times A$ of the product $A \times A$.



(Relations on the set of natural numbers)

A relation R on A is

- often denoted as an infix symbol
  $a \leq b$; $a \not\lessgtr b$; $a \approx b$; $a \downarrow b$
  all mean that $(a, b) \in (\leq), (\not\lessgtr) \ldots$
- called **REFLEXIVE** if $\forall a \in A$, $(a,a) \in R$
- called **TRANSITIVE** if $\forall a, b, c \in A$
  $(a,b) \in R$ and $(b,c) \in R \xrightarrow[\text{IMPLIES}]{} (a,c) \in R$.

A set equipped with a reflexive and transitive relation R is called an ordered set.

Defenition An ORDERED SET $(A, \leq)$ is a pair

$\boxed{A}$ a set

$\boxed{\leq}$ a relation on A denoted as an infix as before

$\leq$ is REFLEXIVE $\forall a \in A : a \leq a$.

is TRANSITIVE $\forall a, b, c \in A : a \leq b, b \leq c \to a \leq c$

The relation on an ordered set is called its order relation

The set A is called the support or carrier of the ordered set

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Examples and nonexamples

The set $\mathbb{N}$ of natural numbers $\{0, 1, 2, \dots\}$ can be equipped with an order relation formalizing (by induction!) the intuitive idea that

$$\left[\, 0 \leq 0 \right.$$

$$\left[\begin{array}{l} 0 \leq 1 \\ 1 \leq 1 \end{array}\right.$$

$$\left[\begin{array}{l} 0 \leq 2 \\ 1 \leq 2 \\ 2 \leq 2 \end{array}\right.$$

$$\vdots$$

(Package this idea in a precise defenition: for example declare that

$$\begin{cases} 0 \leq n & \text{for every } n \\ \text{if } m \leq n & \text{then } m \leq n+1 \\ \text{if } n \leq m & \text{then } n+1 \leq m+1 \end{cases}$$

Another important class of examples is obtained as follows: fix any set X and recall that we can form the set

$$PX = \{ U \subseteq X \}$$

of all subsets of X.

$U \leq V$ if $U \subseteq V$

Then the relation of CONTAINMENT defines an order relation on PX.

- $U \subseteq U$ is obvious $\forall a.\ a \in U \rightarrow a \in U$
- if $U \subseteq V$, $V \subseteq W$ then $U \subseteq W$ (ultimately because logical implication is transitive!)

A 3rd class of examples: divisibility relation

On the set of natural numbers define "$a \mid b$" (read a divides b) as

$b = ka$ for some natural number k.

- REFLEXIVE: $a \mid a$ because $a = 1 \cdot a$
- TRANSITIVE: $a \mid b$, $b \mid c$. Then

$$\begin{aligned} b &= ka \\ c &= hb \end{aligned}$$ for $h, k \in \mathbb{N}$. But then $c = hb = hka$

A few counterexamples follow:

STRICT CONTAINMENT

$U \subsetneq V$ means that $a \in U \to a \in V$ but we rule out the possibility that $U = V$

By construction then $\subsetneq$ is not reflexive.

But this example sounds a bit artificial!

Can you come up with one from "real life"?

Some suggestions:

- x is the father of y ($x, y \in$ set of people)
  Certainly <u>not</u> reflexive.
  <u>not</u> transitive
- x preys on y (x, y in a set of animals)
- x loves y (hopefully reflexive; not transitive)
- x is the successor of y ($x, y \in \mathbb{N}$)
  not reflexive
  not transitive
- x infects y (x, y animals, e.g. x is a virus or a parasite)
- x is perpendicular to y (x, y lines in the plane)
  (think about why it is not transitive...)
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(↑ your <u>example here</u>.) ______