- Antisymmetry; skeletal orders

An ordered set $(A, \leq)$ is called THIN or "skeletal" if the relation $\leq$ satisfies

$$(a \leq b \ \& \ b \leq a) \longrightarrow a = b \qquad (*)$$

- $(\mathbb{N}, \leq)$ is thin
- $(\mathbb{N}, |)$ is thin
- $(PX, \subseteq)$ is thin

$\vdots$

The name is justified in terms of the following small proposition

Prop Let $(A, \leq)$ be a skeletal order. Then there is no nontrivial sequence of distinct elements

$$a \leq a_1 \leq a_2 \leq \dots \leq a_n \leq a$$

In other words for each such sequence, one must have $a_1 = a_2 = \dots = a_n = a$.

Proof Induction on n

- $a \leq a_1 \leq a \to a = a_1$ is just $(*)$ above

- If $a \leq a_1 \leq \_\_ \leq a_n \leq a_{n+1} \leq a$ then

$a \leq a_1 \leq \dots \leq a_n \leq a \to$ by inductive hyp $\underline{\text{all} =}$

$a_n = a \leq a_{n+1} \leq a \to (*)$ says now that $\underline{a_{n+1} = a}$.

Definition Let $(P, \leq)$ be an ordered set

we say that $x \equiv y$ (read x is equivalent to y)

if $x \leq y$ and $y \leq x$.

(Observe that on a skeletal order, this relation is "trivial", which means that $x \equiv y$ if and only if $x = y$)

Observe also that

1. $x \equiv x$ (refl)
2. $x \equiv y \to y \equiv x$ (sym)
3. if $x \equiv y$ and $y \equiv z$ then $x \equiv z$. (trans)

Proof Lets do it together.

$\equiv$ is an example of an equivalence relation, used when one wants to declare elements of a set "the same" even if strictly speaking they are not syntactically equal.

Some examples :

⟨ $x \equiv y$ if both are red : colorwise equivalence

$x \equiv y$ if they are both even numbers equivalence "modulo 2"

$x \equiv y$ if they are both circles (shape wise equivalence) ⟩

EXTENDING ORDERS

Recall: $(PX, \subseteq)$ is an ordered set.

But then when $X = A \times A$ it follows that the set of all relations on A is ordered by $\subseteq$!
In particular the set of all order relations on a set A is an ordered set:

Given a set A we can compare two order relations on A when one is contained in the other as subsets of $A \times A$.

In simpler terms an order $\leq_1$ is contained in an order $\leq_2$ when

$$a \leq_1 b \longrightarrow a \leq_2 b$$

(but there can be elements comparable under $\leq_2$ that are not comparable under $\leq_1$!)

(Cursed notation: $\leq_1 \subseteq \leq_2$)
(but completely formal!)

In particular given an ordered set $(A, \leq)$ I can consider all orders on A extending $\leq$:

$$E(\leq) = \{ \overset{\text{curly}}{\preccurlyeq} \text{ order on } A \mid \overset{\text{straight}}{\leq} \subseteq \overset{\text{curly}}{\preccurlyeq} \}$$

Szpilrajn's theorem

Given $(A, \leq)$ ordered set, $E(\leq)$ contains always at least a <u>total</u> order

(A total order on A is an order relation $\lesssim$ such that "all elements can be compared", or formally given $a, b \in A$

$$a \leq b \quad \text{or} \quad b \leq a$$

(either $(a,b) \in R$ or $(b,a) \in R$)

Examples: $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$

Nonexamples: $(PX, \subseteq)$, $(\mathbb{N}, -|-)$

(The proof goes beyond our current scope but if you want some food for thought, try to prove S. theorem when A is finite ...)

(for infinite A you need to use a magic formule called "axiom of choice")

[Given $(A, \leq)$ it's either $x \leq y$, in which case you do nothing, or $(x,y) \notin \leq$ in which case you add (y,x) to $\leq$. <u>Problem</u>: this destroys antisymmetric property ...]