TOPS AND BOTTOMS

Def Let $(P, \leq)$ be an ordered set.
A TOP ELEMENT, if it exists, is an element $\top \in P$ such that

$$\forall a \in P, \quad a \leq \top$$

A BOTTOM ELEMENT, if it exists, is $\bot \in P$

$$\forall a \in P, \quad \bot \leq a$$

Examples and non examples

- $(\mathbb{N}, \leq)$ has bottom 0, not top
- $(\mathbb{Z}, \leq)$ has no top, no bottom (it is "unbounded")
- $(PX, \subseteq)$ has bottom $(\emptyset)$ and top (all X)
- The finite total order $\{0 \leq 1 \leq \dots \leq n\}$ has top and bottom
- $\left\{ \begin{array}{ccc} x \leq & & \leq b \\ & a & \\ y \leq & & \leq c \end{array} \right\}$ has no top, no bottom

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Proposition If $\top_1, \top_2$ are both top elements in $(P, \leq)$ then $\top_1 = \top_2$

In particular if $(P, \leq)$ is a thin order there exists at most one top element

Proof $\top_1 \leq \top_2$ because $\top_2$ is top
$\top_2 \leq \top_1$ because $\top_1$ is top. $\blacksquare$

THE OPPOSITE ORDER

Let $(P, \leq)$ be an ordered set, define a new order on the same set of elements denoted $\leq^{op}$ or (rather conveniently) $\geq$ declaring that $\underline{a \leq^{op} b}$ iff $\underline{b \leq a}$ in P.

Easy exercise for you: prove $\geq$ is also an order; prove $\leq^{op}$ is thin if and only if $\leq$ is thin.

1) The opposite order for $(PX, \subseteq)$ is "being a superset"
2) The opposite order for $(\mathbb{N}, \leq)$ is

$$\{ \ldots \leq 3 \leq 2 \leq 1 \leq 0 \}$$

Top for $\supseteq$ in PX : $\emptyset$
Bottom for $\supseteq$ in PX : X

Top for $\geq$ in $\mathbb{N}$: 0
Bottom $\geq$ $\mathbb{N}$: does not exist

Fact Easy exercise for zealous students

$(P, \leq)$ has top $\top \leftrightarrow (P, \geq)$ has bot $\top$

$(P, \leq)$ has bot $\bot \leftrightarrow (P, \geq)$ has top $\bot$

So if you wonder why all this fuss about the opposite order, here's why:

- Every property $\mathcal{P}$ that can be stated in an order $(P, \leq)$ admits a dual $\mathcal{P}^{op}$ where "the direction of all $\leq$'s have been reversed", and
- $\mathcal{P}$ is true for a poset $(P, \leq)$ if and only if $\mathcal{P}^{op}$ is true for $(P, \geq)$.

This falls under the name of

PRINCIPLE OF DUALITY

in order theory

Every "$\leq$-statement" that can be formulated in the language of order theory can be DUALIZED into its correspondent "$\geq$-statement", and the strategy to prove the $\geq$-statement is precisely the $\leq$-strategy, up to replacing every $\leq$ with $\geq$.

MONOTONE FUNCTIONS

The idea of monotone functions (and for that matter of every notion of "homomophism" between structures) is to be a map

$$f : X \longrightarrow Y$$

between structured sets that "preserves" said structure.

In the case X, Y have order relations on them, "preserving the structure" means preserving the precedence relation established by the order: a MONOTONE function is $f : X \to Y$

If $a \leq b$ in X, then $f(x) \leq f(y)$ in Y.

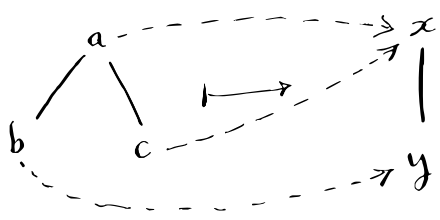
Examples:

0) The identity, of course

1) $(\mathbb{N}, \leq) \longrightarrow (\mathbb{N}, \leq) : n \longmapsto n+1$

$n \longmapsto 2^n$

2)



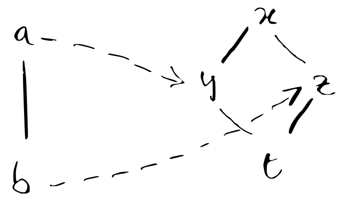
3) If $f : X \to Y$, $g : Y \to Z$ are both monotone, then their composite is monotonic

4) $PX \xrightarrow{C_A} PX$ is it monotone?

$U \longmapsto C_A(U) = A \cup U$

4') $PX \longrightarrow PA : U \longmapsto U \cap A$ monotone?

5) A non example



6) $(\mathbb{N}, |) \longrightarrow (\mathbb{N}, \leq)$

is the identity function monotone?

- $f : (X, \leq) \longrightarrow (Y, \leq)$ does it have to preserve the top/bottom element?
  (A: No; we say that $f : X \to Y$ preserves top/bottom when it does)

- $f : (X, \leq) \longrightarrow (Y, \leq)$ monotone btwn orders
  then $x \equiv x'$ in $X \Rightarrow fx \equiv fx'$ in Y.

Proof Let's do it together!

Def (Top extension of an order)
Let $(P, \leq)$ be an order; define $P^{\triangleright}$ as the set $P \cup \{\infty\}$ where $\infty \notin P$.
$P^{\triangleright}$ admits an order $\leq^*$ "forcing ∞ to be the top"

$a \leq^* b$ true | for every a if $b = \infty$
| if $a \leq b$ in P.