SUPREMA/INFIMA

Let $(P, \leq)$ be an ordered set, $S \subseteq P$ a subset an infimum for S also called meet, or greatest lower bound is an element $y \in P$ such that

1) $y \leq s$ for every $s \in S$
2) if $z \leq s$ for every $s \in S$, then $z \leq y$

NOTATION: $y = \bigwedge S$ or $y = \inf S$

Remark Any two infima of the same S are $\equiv$- equivalent

Proof Usual trick. ∎

Corollary If $(P, \leq)$ is a thin order $\inf S$ is unique when it exists (it can fail to exist, for example...)

Inf S for $S \subseteq P^{op}$ are called suprema (or joins, or least upper bounds)

1) $s \leq y$ for all $s \in S$
2) If $s \leq z$ for all $s \in S$, then $y \leq z$.

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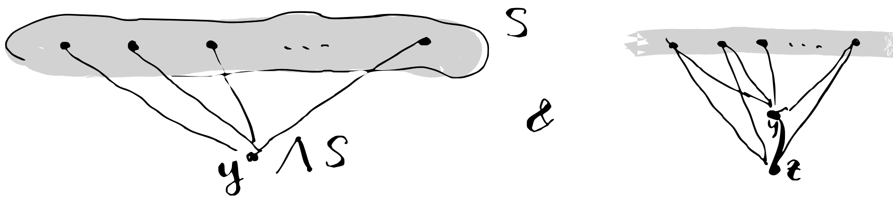
From now on we work in a thin order
So whenever we write "$\inf S$" we are sure it's unique.

- Representing sup/inf in the Hasse diagram of P



- If $S = \{x, y\}$ has an inf we denote $\bigwedge\{x,y\}$ as an infix operation $x \wedge y$

▷ We can extend the assignment $\{x,y\} \longmapsto \bigwedge\{x,y\}$ to a function $P \times P \longrightarrow P$ w/ the observation that $x \wedge x = \bigwedge\{x,x\} = \bigwedge\{x\} = x$
This property of a binary operation is called IDEMPOTENCY.

Observe that

- $\wedge$ is commutative
- $\wedge$ is associative
- If P has a top element $\top$, $\top \wedge x = x$ for every $x \in P$

In order to prove the last property we have to prove this first:

If S is empty and $\bigwedge S$ exists in P, then $\bigwedge S =$ top element (call it t)

Proof

1) $\forall s \in S \quad t \leq s$ ☑ $(S = \emptyset)$

2) $\underbrace{\underbrace{(\forall s \in S \;\; z \leq s)}_{\text{vuota}} \longrightarrow z \leq t}_{\text{vera}}$ ☑

(At this point you might feel confused, so let's start a brief

digression

on VACUOUS TRUTH.

(Ex falso quodlibet sequitur)

FROM [WHAT IS] FALSE, EVERYTHING FOLLOWS.

VALEST (ELATIVE) JÄRELDUB (3rd SING) KÕIKE (PART)

In the next page, proof of associativity..

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$

<u>claim</u> $\quad x \wedge (y \wedge z) = (x \wedge y) \wedge z$

$\qquad\qquad$ LHS $\qquad\qquad$ RHS

LHS $= \bigwedge \{x, y \wedge z\}$ satisfies

1) LHS $\leq x$

2) LHS $\leq y \wedge z$

From 2) $\quad$ LHS $\leq y$, LHS $\leq z$ (trans) ; but then

$$\begin{cases} \text{LHS} \overset{\text{UP}}{\leq} x \wedge y \\ \text{LHS} \leq z \end{cases} \xrightarrow[\text{UP}]{} \text{LHS} \leq \bigwedge\{x \wedge y, z\} = \text{RHS}$$

Similarly

RHS $= \bigwedge \{x \wedge y, z\}$ satisfies

1) RHS $\leq x \wedge y$

2) RHS $\leq z$

From 1), $\quad$ RHS $\leq x$, RHS $\leq y$ (transitivity)

but then

$$\begin{matrix} \text{RHS} \leq x \\ \text{RHS} \underset{\text{UP}}{\leq} y \wedge z \end{matrix} \overset{\text{UP}}{\Longrightarrow} \text{RHS} \leq \bigwedge\{x, y \wedge z\} = \text{LHS}.$$

$\blacksquare$

(Exercise: dualize everything for $\sup\{x, y\}$)

From the completely algebraic structure given by $\_\wedge\_$ it is possible to recover the order on $\mathbb{P}$:

Theorem $(P, \leq)$ a thin order. Then the following conditions are equivalent

1) $x \leq y$
2) $x \wedge y = x$
3) $x \vee y = y$

provided $x \wedge y$, $x \vee y$ exist.

Proof [...]