# MONOIDS

**Idea** In a monoid, elements can be concat'd associatively, and there is an element the concat with which has no effect.

Examples of monoids are

- natural numbers w/ sum
  product
- integers with sum, or product
- $P(X)$ where $U * V = U \cap V$
- $P(X)$ where $U * V = U \cup V$
- Lists in an alphabet
- The set of functions $A \to A$ ($A$ any set) with function composition.
- ... and many others

**Definition** A monoid is a set $M$ equipped w

- An operation $M \times M \xrightarrow{m} M$ (denoted infix)
- an element $e \in M$

such that

$$\begin{cases} a \cdot (b \cdot c) = (a \cdot b) \cdot c & \forall abc \\ a \cdot e = e \cdot a = a & \forall a \end{cases}$$

Verify the previous ones are all examples. [ Done together ]

Find <u>non</u>-examples:

- A set $X$ with a binary operation not closed ($a, b \in X$, $a \cdot b \notin X$)
- a set $Y$ with a closed bin op not associative.
- a set $Z$ with a closed, assoc bin op but without identity element

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Classes of monoids

- A monoid is COMMUTATIVE if ...
- A monoid is a GROUP if every $m \in M$ has an element $\bar{m}$ such that $m\bar{m} = \bar{m}m = e$.
- Non commutative monoids are $(A^A, \circ, id)$
- Non-group monoids are $(PX, \cap, X)$
  $(\mathbb{N}, +)$
  $\vdots$

<u>Exercise</u>. Let $M$ be a monoid. $P(M)$
its powerset $(M, \cdot, e)$

Define an operation $\circledast$ on $PM$ as

$$U \circledast V = \{ u \cdot v \mid u \in U, v \in V \}$$

Is $\circledast$ associative? Is $PM$ a monoid?
If $M$ commutative, $PM$ commutative?
(Prove, or find counterexample).

<u>Submonoids</u>:

A SUBMONOID of a monoid $(M, \cdot, e)$
is a subset $N \subseteq M$ such that

- $e \in N$
- $a, b \in N \Rightarrow a \cdot b \in N$.

- Examples of submonoids $\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\}$ Among the examples we have
- Non examples of submonoids
  - $PA \subseteq (PX, \cap)$ $(X \notin PA)$
  - $\left\{ \text{Find some among the examples we have} \right\}$

(e.g. "lists of length $\leq 7$")

# Monoid homomorphisms

Idea: have operations
want to preserve them

Given monoids $M, N$

$$f : (M, *, e) \longrightarrow (N, \cdot, u)$$

a function $f : M \to N$ is a homomorphism

if

$$f(m_1 * m_2) = f(m_1) \cdot f(m_2)$$

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Examples and non-examples

$$(\mathbb{N}, +) \xrightarrow{3^{(-)}} (\mathbb{N}, \cdot) \ ?$$

$$(PX, \cap) \xrightarrow{(-)^c} (PX, \cup) ? \text{ complement}$$

$$(\mathbb{Z}, +) \xrightarrow{2\cdot(-)} (\mathbb{Z}, +)$$

Non ex:

- Does not preserve 1 ________
- Does not preserve the operation ________

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If still time, talk about lists ...