A short recap on products of monoids

· Recall the def of monoid homomorphism btwn two monoids M, N:
a function $f: M \longrightarrow N$ such that

$$\boxed{f(m \underset{\substack{\uparrow \\ \text{operation in } M}}{\cdot} m') = f(m) \underset{\substack{\uparrow \\ \text{op'n in } N}}{\cdot} f(m')}$$

Definition ISOMORPHISM of monoids $M \xrightarrow{f} N$
$N \xrightarrow{g} M$
$fg = id;\ gf = id$

+ we have seen a few examples.

+ Defined the product of monoids as

$$M \times N = \{(m, n) \mid m \in M, n \in N\}$$

with operation pointwise
identity $(1_M, 1_N)$.

(!) A central idea of category theory is that one can recognize $M \times N$ thru a UNIVERSAL PROPERTY:

A monoid P is $M \times N$ IF AND ONLY IF

P satisfies a certain property that will characterize it "uniquely"

(Similarly, "being a top/bottom" characterizes "unique")

Start observing that for $M \times N$ the following property is true (*)

« Given a monoid A and monoid homomorphisms $u : A \longrightarrow M$
$v : A \longrightarrow N$

there exists a unique monoid homomorphism $\langle u,v \rangle : A \longrightarrow M \times N$

with the property that

$$u = \left(A \xrightarrow{\langle u,v \rangle} M \times N \xrightarrow{\mathrm{proj}_M} M \right)$$

$$v = \left(A \xrightarrow{\langle u,v \rangle} M \times N \xrightarrow{\mathrm{proj}_N} N \right)$$ »

- Define $\langle u,v \rangle$
- Show that if there is another $\langle u,v \rangle'$
- with the same property then $\langle u,v \rangle = \langle u,v \rangle'$

<u>CLAIM</u> If P satisfies the same property then there is a <u>unique</u> [bijective homomorphism] $P \xrightarrow{\sim} M \times N$.

isomorphism

So, P is "essentially equal" to $M \times N$

Start by studying what happens when $A = M \times N$ in the previous property (*)

Diagrammatic notation:

$$\begin{tikzcd} & A \arrow[dl, "u"'] \arrow[d, "!\ \langle u,v\rangle"] \arrow[dr, "v"] & \\ M & M\times N \arrow[l, "proj_M"] \arrow[r, "proj_N"'] & N \end{tikzcd}$$

& if P satisfies the same property $P \simeq M\times N$

<u>Proof</u>: Since P satisfies the U.P.

$\exists!\ M\times N \xrightarrow{k} P$ such that

$$\begin{tikzcd} & M\times N \arrow[dl, "proj_M"'] \arrow[d, "!\ k"] \arrow[dr, "proj_N"] & \\ M & P \arrow[l, "p_M"] \arrow[r, "p_N"'] & N \end{tikzcd}$$

Since $M\times N$ satisfies the U.P.

$\exists!\ P \longrightarrow M\times N$

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_M"'] \arrow[d, "!\ h"] \arrow[dr, bend left, "p_N"] & \\ M & M\times N \arrow[l, "proj_M"] \arrow[r, "proj_N"'] & N \end{tikzcd}$$

Uniqueness implies that $P \to M\times N \to P = id$

$M\times N \to P \to M\times N = id$

(because if $A = P$, id does the job

if $A = M\times N$, id does the job).

More precisely: if $A = P$

$$
\begin{tikzcd}
 & P \arrow[dl] \arrow[d, equal] \arrow[dr] & \\
M & P \arrow[l] \arrow[r] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P \arrow[ddl] \arrow[d, "h"] \arrow[ddr] & \\
 & M \times N \arrow[d, "k"] & \\
M & P \arrow[l] \arrow[r] & N
\end{tikzcd}
$$

both solve the problem

If $A = M \times N$

$$
\begin{tikzcd}
 & M \times N \arrow[dl] \arrow[d, equal] \arrow[dr] & \\
M & M \times N \arrow[l] \arrow[r] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M \times N \arrow[ddl] \arrow[d, "k"] \arrow[ddr] & \\
 & P \arrow[d, "h"] & \\
M & M \times N \arrow[l] \arrow[r] & N
\end{tikzcd}
$$

both solve the problem.

But then, $kh = id$ & $hk = id$

Morale of this story

Category theory is a language apt to characterize structures and operations on structures (like Cartesian products) "synthetically", which means without looking inside the carriers of the structures, but relying <u>only</u> on universal properties of arrow diagrams.

FREE MONOIDS

Definition of the monoid of lists

Let A be a set, consider the set

$$List(A) = \{(a_1 \_ a_n) \mid a_i \in A, n \geq 0\}$$

with the convention that if $n = 0$

$()$ is the empty list, with no elements

Constructors for $List(A)$:

List A :

$cons(a, as) = a :: as$

[] : List A

cons : $A \to List\ A \to List\ A$

-- cons (a, as) generates the list

-- having a as head, as as tail

Monoid operation on $List(A)$: concat

$(++) : List\ A \to List\ A \to List\ A$

$[] ++ ys = ys$

$(x :: xs) ++ y = x :: (xs ++ ys)$

This is a monoid operation

- – associative
- – [] is identity on both sides

$\Big\}$ proof by induction

The monoid List(A) is clearly non-commutative

The monoid List A has no nontrivial invertible elements

(In a monoid M, $x \in M$ is INVERTIBLE if $\exists y$ such that $xy = 1$, $yx = 1$)

$$as ++ bs = [] \implies as = bs = []$$

<u>Proposition</u> Let $A = \{s\}$ a set with a single symbol.

Then List(A) is the monoid of natural numbers

<u>Proof</u> Define $\mathbb{N}$ as the set

$$\{0, s0, ss0, sss0, ssss0, \ldots\}$$

List A, (otoh), is the set

$$\{[], (s), (ss), (sss), (ssss), \ldots\}$$

And these two sets are evidently "the same" up to relabeling their elements _