

A short recap on products of monoids

Recall the def of monoid homomorphism
btwn two monoids M, N :

a function $f: M \rightarrow N$ such that

$$\boxed{f(m \cdot m') = f(m) \cdot f(m')}$$

\uparrow operation in M \uparrow op'n in N

Definition ISOMORPHISM of monoids

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ N & \xrightarrow{g} & M \\ fg = \text{id} & ; & gf = \text{id} \end{array}$$

+ we have seen a few examples -

+ Defined the product of monoids as

$$M \times N = \{ (m, n) \mid m \in M, n \in N \}$$

with operation pointwise

identity $(1_M, 1_N)$ -

⚠ A central idea of category theory.
is that one can recognize $M \times N$
thru a UNIVERSAL PROPERTY:

A monoid P is $M \times N$ IF AND ONLY IF

P satisfies a certain property that
will characterize it "uniquely"

(Similarly, "being a top bottom" characterizes "uniquely")

Start observing that for $M \times N$ the following property is true $(*)$

\ll Given a monoid A and monoid homomorphisms $u: A \rightarrow M$
 $v: A \rightarrow N$
 there exists a unique monoid homomorphism $\langle u, v \rangle: A \rightarrow M \times N$
 with the property that
 $u = (A \xrightarrow{\langle u, v \rangle} M \times N \xrightarrow{\text{proj}_M} M)$
 $v = (A \xrightarrow{\langle u, v \rangle} M \times N \xrightarrow{\text{proj}_N} N) \gg$

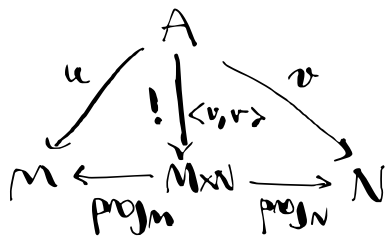
- Define $\langle u, v \rangle$
- Show that if there is another $\langle u, v' \rangle$
- with the same property then $\langle u, v \rangle = \langle u, v' \rangle$

CLAIM If $P \simeq$ satisfies the same property then there is a unique [bijective homomorphism] $P \xrightarrow{\simeq} M \times N$.

So, P is "essentially equal" to $M \times N$

Start by studying what happens when $A = M \times N$ in the previous property $(*)$

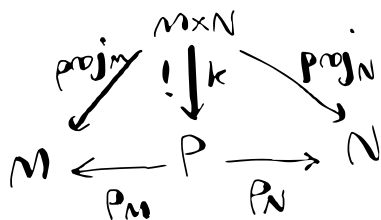
Diagrammatic notation:



& if P satisfies
the same
property $P \simeq M \times N$

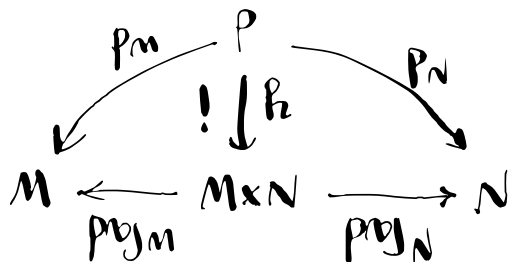
Proof: Since P satisfies the U.P.

$\exists!$ $M \times N \xrightarrow{k} P$ such that



Since $M \times N$ satisfies the U.P.

$\exists!$ $P \longrightarrow M \times N$



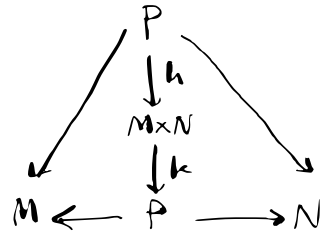
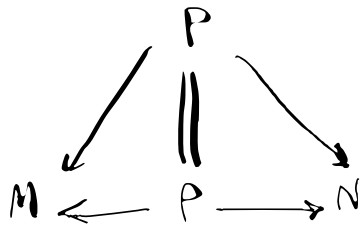
Uniqueness implies that $P \rightarrow M \times N \rightarrow P = \text{id}$

$M \times N \rightarrow P \rightarrow M \times N = \text{id}$

(because if $A = P$, id does the job

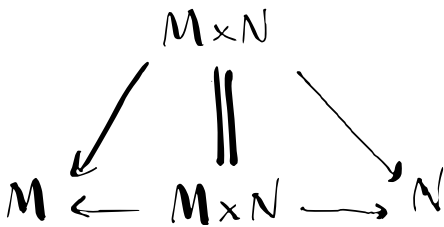
if $A = M \times N$, id does the job)-

More precisely: if $A = P$



both solve the problem

If $A = M \times N$



both solve the problem.

But then, $kh = id$ & $hk = id$

Morale of this story

Category theory is a language apt to characterize structures and operations on structures (like Cartesian products) "synthetically", which means without looking inside the carriers of the structures, but relying only on universal properties of arrow diagrams.

FREE MONOIDS

Definition of the monoid of lists

Let A be a set, consider the set

$$\text{List}(A) = \{(a_1 \dots a_n) \mid a_i \in A, n \geq 0\}$$

with the convention that if $n=0$
() is the empty list, with no elements

Constructors for $\text{List}(A)$:

$\text{List } A$:

$[] : \text{List } A$

$$\text{cons}(a, as) = a :: as$$

$\text{cons} : A \rightarrow \text{List } A \rightarrow \text{List } A$

-- $\text{cons}(a, as)$ generates the list

-- having a as head, as as tail

Monoid operation on $\text{List}(A)$: concat

$(++) : \text{List } A \rightarrow \text{List } A \rightarrow \text{List } A$

$$[] ++ ys = ys$$

$$(x :: xs) ++ y = x :: (xs ++ y)$$

This is a monoid operation

• - associative

• - $[]$ is identity on both sides

} proof
by
induction

The monoid $\text{List}(A)$ is clearly non-commutative

The monoid $\text{List}A$ has no nontrivial invertible elements

(In a monoid M , $x \in M$ is INVERTIBLE if $\exists y$ such that $xy = 1$, $yx = 1$)

$$as + bs = [] \implies as = bs = []$$

Proposition Let $A = \{s\}$ a set with a single symbol.

Then $\text{List}(A)$ is the monoid of natural numbers

Proof Define \mathbb{N} as the set

$\{0, s0, ss0, sss0, ssss0, \dots\}$
List A , $\{()\}$, is the set
 $\{[], (s), (ss), (sss), (ssss), \dots\}$

And these two sets are evidently

"the same" up to relabeling their elements —