Tallinn University of Technology

Category Theory and its Applications – ITI9200

Exercise sheet 1

assigned: February 26, 2023
due: March 8, 2023

Exercise 1:
Let X denote a set, and find a relation $R \subseteq X \times X$ that is

- reflexive and symmetric, but not transitive;
- symmetric and transitive, but not reflexive;
- reflexive and transitive, but not symmetric;
- reflexive, but neither symmetric nor transitive;
- symmetric, but neither reflexive nor transitive;
- transitive, but neither reflexive nor symmetric;
- not symmetric, not transitive, not reflexive.

(You can choose different sets X for each item of the list; but additional points if you manage to find the same X with a different relation on it for each item.)

Exercise 2:

(In this exercise, assume all ordered sets are thin, i.e. the order relation is antisymmetric.) Recall that a *Galois connection* between two ordered sets P, Q consists of a pair $f_* : P \leftrightarrows Q : f^*$ of monotonic functions with the property that

$$f_* p \leq q \iff p \leq f^* q.$$

The map f_* is called the *left part* of the Galois connection, and f^* is called the *right part.*

- Prove that there is a unique monotone function $!_P : P \to [1]$ (read: 'bang P'), where $[1]$ is the ordered set having only one point $\{*\}$, and the obvious order relation $* \leq *$ is the only possible choice.

- Prove that P has a top element if and only if $!_P$ is the left part of a Galois connection; dually, prove that P has a bottom element if and only if $!_P$ is the right part of a Galois connection.

- Let $f_* : P \leftrightarrows Q : f^*$ be a Galois connection; prove the *Galois identities*

$$\forall p \in P,\ \ f_* f^* f_*(p) = f_*(p) \qquad \forall q \in Q,\ \ f^* f_* f^*(q) = f^*(q).$$

Exercise 3:

(On down-sets.) Let $(P, \leq)$ be a thin poset; a *down-set* S of P is a subset $S \subseteq P$ such that if $x \in S$ and $y \leq x$, then $y \in S$. Let DP denote the set of all down-sets of P, ordered by inclusion; for each $x \in P$, define the downset ${\downarrow}x$ *generated* by x as the set $\{y \in P \mid y \leq x\}$.

- Show that the map $x \mapsto {\downarrow}x$ is a monotone and injective function ${\downarrow}(-) : P \to DP$, i.e. show that
$$a \leq b \Rightarrow {\downarrow}a \leq {\downarrow}b; \qquad {\downarrow}a = {\downarrow}b \Rightarrow a = b$$

- Let $(P, \leq)$ admit suprema for all down-sets; show that $S \mapsto \sup S$ defines a monotone map $DP \to P$, and show that
$$\sup S \leq x \iff S \subseteq {\downarrow}x$$
for every $S \in DP$ and $x \in P$. Compare $(\sup S)^{\downarrow}$ and S: who is contained in the other? Compare $\sup(x^{\downarrow})$ and x, who is bigger?

- Define ${\downarrow}U := \bigcup_{x \in U} {\downarrow}x$ for every $U \subseteq P$. Prove that U is a down-set if and only if ${\downarrow}U = U$. Is it true that ${\downarrow}{\downarrow}U = {\downarrow}U$?

Exercise 4:

(Algebraic and order lattices.) An *algebraic lattice* $(X, \wedge, \vee)$ is a set X equipped with binary operations $\wedge, \vee$ enjoying the following properties: for all $a, b, c \in X$,

- (commutative) $a \wedge b = b \wedge a$ e $a \vee b = b \vee a$;
- (associative) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$;
- (absorption laws) $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Given this,

1. Prove that from the absorption laws it follows that both $\wedge$ and $\vee$ are *idempotent* operations: for all $a \in X$, one has
$$a \wedge a = a \qquad a \vee a = a.$$
2. If $(X, \wedge, \vee)$ is an algebraic lattice, we can define a thin order relation on X by defining $a \leq^{\mathrm{a}} b$ if $a \wedge b = a$. Prove that $a \leq^{\mathrm{a}} b$ if and only if $a \vee b = b$; prove that for every $a, b, c \in X$ the following inequalities hold

   D1) $(a \wedge b) \vee (a \wedge c) \leq^{\mathrm{a}} a \wedge (b \vee c)$;

   D2) $a \vee (b \wedge c) \leq^{\mathrm{a}} (a \vee b) \wedge (a \vee c)$.
3. Prove that $(X, \leq^{\mathrm{a}})$ has the property that every two elements $x, y \in X$ have a sup (and this sup is precisely $x \vee y$) and an inf (precisely $x \wedge y$).
4. Prove that $(X, \leq^{\mathrm{a}})$ has a top element if and only if there exists an element $t \in X$ such that $x \wedge t = t \wedge x = x$ for every $x \in X$; dually, $(X, \leq^{\mathrm{a}})$ has a bottom element if and only if there exists $b \in X$ such that $x \vee b = b \vee x = x$ for every $x \in X$.
5. Let $n \geq 1$ be a natural number, and let $[n]$ denote the finite set $\{1, \dots, n\}$ with n elements; call $P[n]$ the set of all subsets of $[n]$. From the previous points it follows that $(P[n], \cup, \varnothing)$ is a commutative monoid. How many monoid homomorphisms are there, of type
$$\varphi : (P[n], \cup, \varnothing) \to (\mathbb{N}, +, 0)$$
(hint: every element U of $P[n]$ is such that $U \cup U = U$, so...)?