Formal Model Theory & Higher Topology

Ivan Di Liberti

TallCat seminar
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Czech Academy of Sciences

This talk is based on three preprints.

1. **General facts on the Scott Adjunction**, ArXiv:2009.14023.
2. **Towards Higher Topology**, ArXiv:2009.14145.
3. **Formal Model Theory & Higher Topology**, ArXiv:2010.00319.

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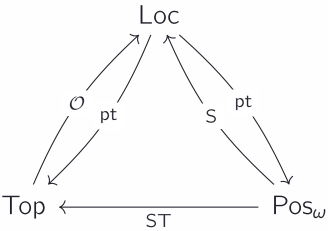
We will start our tour from the crispiest one: (Higher) Topology.

The topological picture



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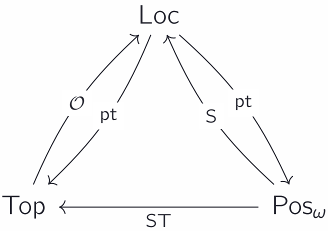

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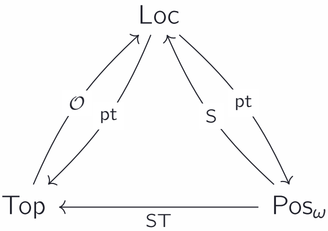

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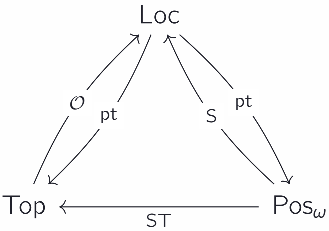

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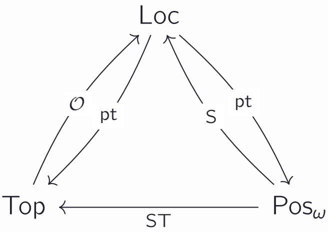

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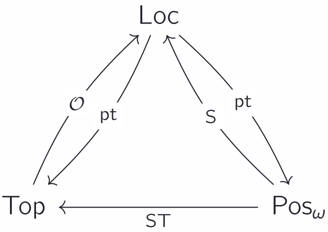

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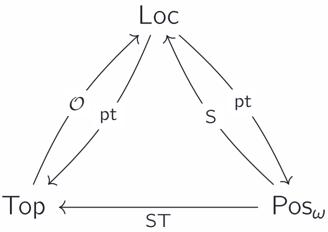

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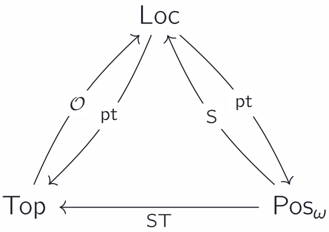

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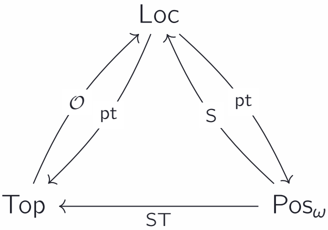

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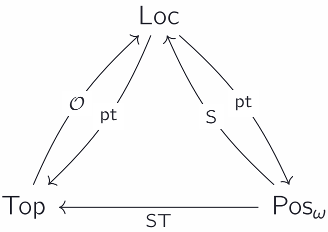

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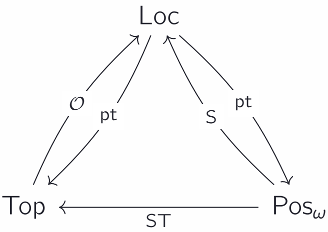

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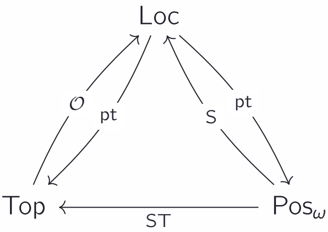

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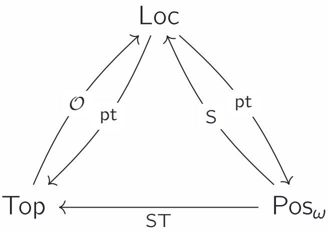

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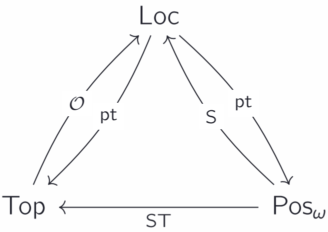

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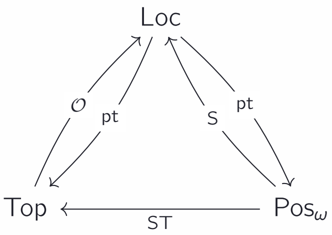

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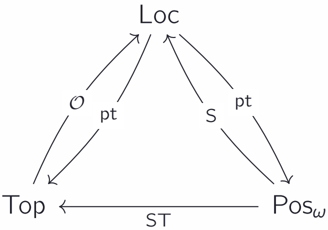

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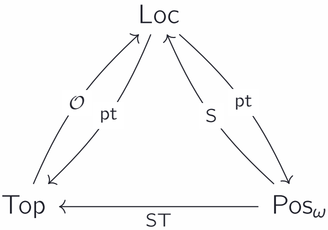

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- S maps a poset with directed colimits to the frame $\mathrm{Pos}_\omega(P, \mathbb{T})$.
- ST equips a poset P with its Scott topology, which can be essentially identified with the frame above.

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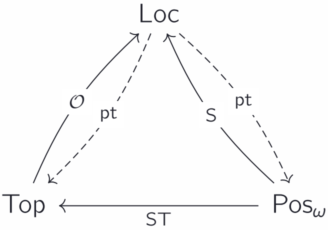

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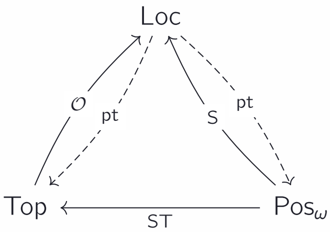

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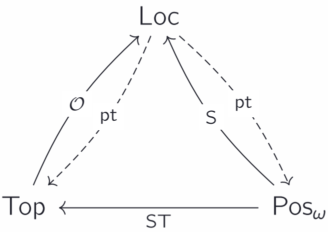

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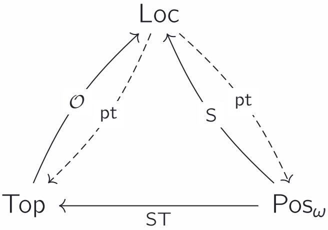

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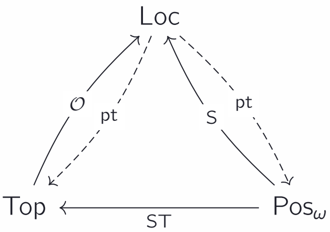

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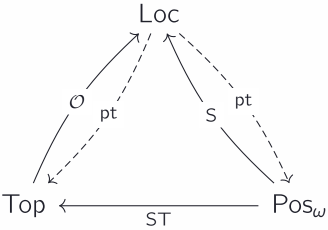

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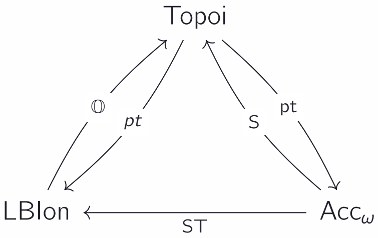

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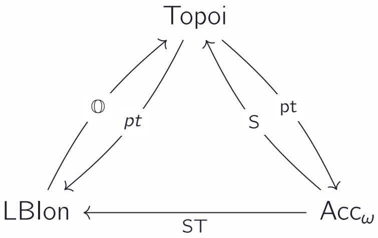

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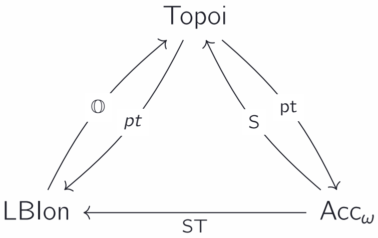

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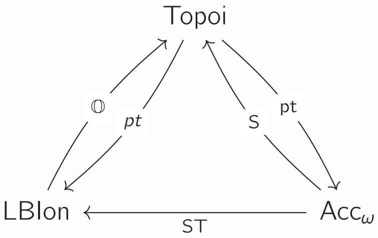

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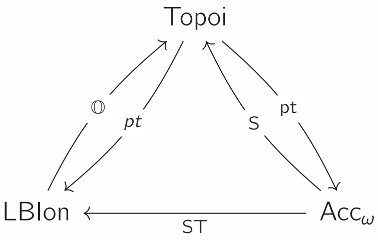

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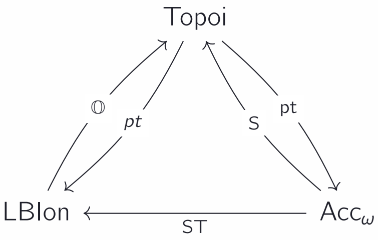

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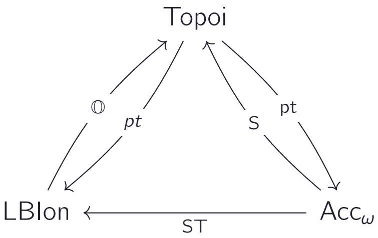

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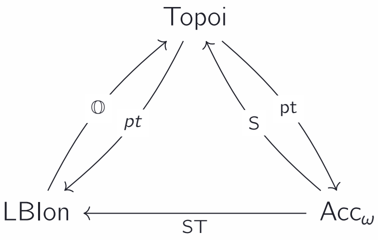

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Ionads

The 2-category of Ionads was introduced by Garner. A **ionad** $\mathcal{X} = (X, \mathsf{Int})$ is a set X together with a comonad $\mathsf{Int} : \mathsf{Set}^X \to \mathsf{Set}^X$ preserving finite limits. While topoi are the categorification of locales, Ionads are the categorification of the notion of topological space, to be more precise, Int categorifies the interior operator of a topological space.

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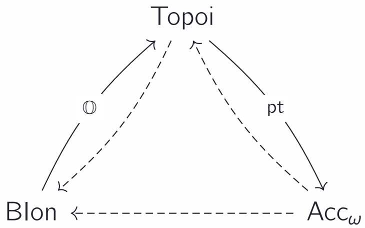
Thm. (Garner)

The category of coalgebras for a ionad is indicated with $\mathbb{O}(\mathcal{X})$ and is a cocomplete elementary topos. A ionad is bounded if $\mathbb{O}(\mathcal{X})$ is a Grothendieck topos. Thus one should look at the functor

$$\mathbb{O} : \mathsf{BIon} \to \mathsf{Topoi},$$

as the categorification of the functor that associates to a space its frame of open sets.


Topoi
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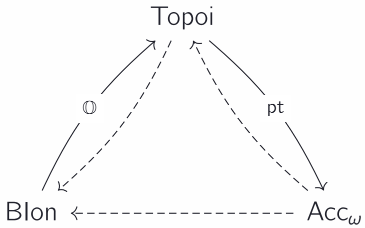
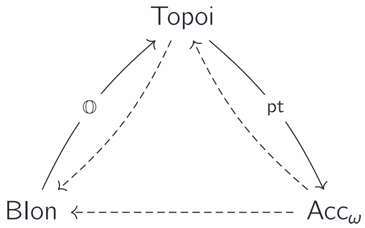


1. The functor pt was also known to the literature. For every topos $\mathcal{E}$ one can define its category of points to be $\mathsf{Topoi}(\mathsf{Set}, \mathcal{E})$, and it is a classical result that this category is accessible and has directed colimits.



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2. My task was to provide all the dashed arrows in this diagram, to show that they form adjunctions and to describe their properties.

The Scott Adjunction (Henry, DL)

There is an 2-adjunction

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2. Topoi is the 2-category of Groethendieck topoi. A 1-cell is a geometric morphism and has the direction of the right adjoint. 2-cells are natural transformation between left adjoints.

The Scott construction

Let $\mathcal{A}$ be a 0-cell in Acc_ω. $\mathsf{S}(\mathcal{A})$ is defined as the category $\mathsf{Acc}_\omega(\mathcal{A}, \mathsf{Set})$.

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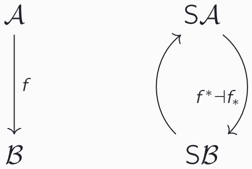
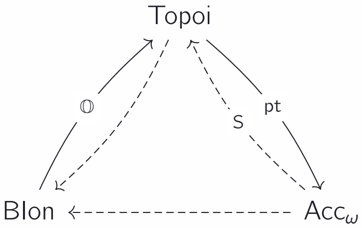
Let $\mathcal{A}$ be a 0-cell in Acc_ω. $\mathsf{S}(\mathcal{A})$ is defined as the category $\mathsf{Acc}_\omega(\mathcal{A}, \mathsf{Set})$.Let $f : \mathcal{A} \to \mathcal{B}$ be a 1-cell in Acc_ω.



$\mathsf{S}f = (f^* \dashv f_*)$ is defined as follows: f^* is the precomposition functor $f^*(g) = g \circ f$. This is well defined because f preserve directed colimits. f^* preserve all colimits and thus has a right adjoint, that we indicate with f_*. Observe that f^* preserve finite limits because finite limits commute with directed colimits in Set.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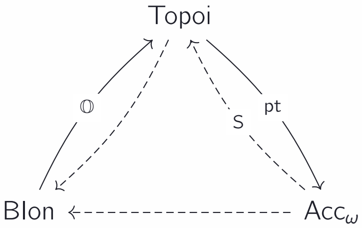

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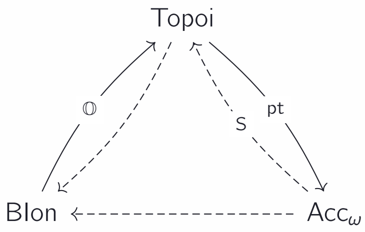

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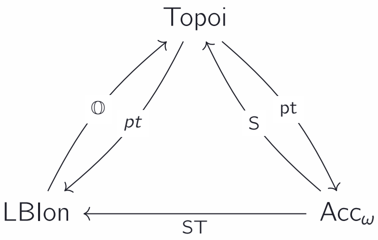
   In order to fix this problem, one needs to stretch Garner's definition and introduce **large (bounded) Ionads**.

Thm. (DL)

Replacing bounded Ionads with large bounded Ionads, there exists a right adjoint for $\mathbb{O}$ and a Scott topology-construction ST such that $\mathsf{S} = \mathbb{O} \circ \mathsf{ST}$, in complete analogy to the posetal case.



The generalized Isbell adjunction (DL)

There is a 2-adjunction

$$\mathbb{O} : \mathsf{LBIon} \leftrightarrows \mathsf{Topoi} : \mathsf{pt}.$$

moreover, the adjunction is idempotent and restrict to a bi-equivelence between sober bounded ionads and topoi with enough points.

Our geometric picture is completed. We now move to a **categorical understanding of the Scott adjunction**.

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Thm.

$\mathrm{Acc}_\omega(\mathcal{A}, \mathcal{B})$ is an accessible category with directed colimits. Thus Acc_ω has an internal hom.

Thm. (DL)

Acc_ω is monoidal closed $(\otimes, \mathrm{Acc}_\omega(-,-))$ with respect to this internal hom.

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The 2-category of topoi is enriched over the bicategory Acc_ω. Moreover it has tensors.

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Thm. (Rosicky, Beke, Lieberman)

A category $\mathcal{A}$ is equivalent to an abstract elementary class iff:

1. it is an accessible category with directed colimits;
2. every map is a monomorphism;
3. it has a *structural* functor $U : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is finitely accessible and U is iso-full, nearly full and preserves directed colimits and monomorphisms.

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Quite not what we were looking for, uh?!

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1. Have a **conceptual understanding** of those accessible categories in which model theory blooms naturally.

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1. Have a **conceptual understanding** of those accessible categories in which model theory blooms naturally.
2. When an accessible category with directed colimits admits such a nice forgetful functor?

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This problem was originally proposed by Rosicky in his talk “Towards categorical model theory” at the 2014 category theory conference in Cambridge: *Show that the category of uncountable sets and monomorphisms between cannot be obtained as the category of point of a topos. Or give an example of an abstract elementary class that does not arise as the category points of a topos.*